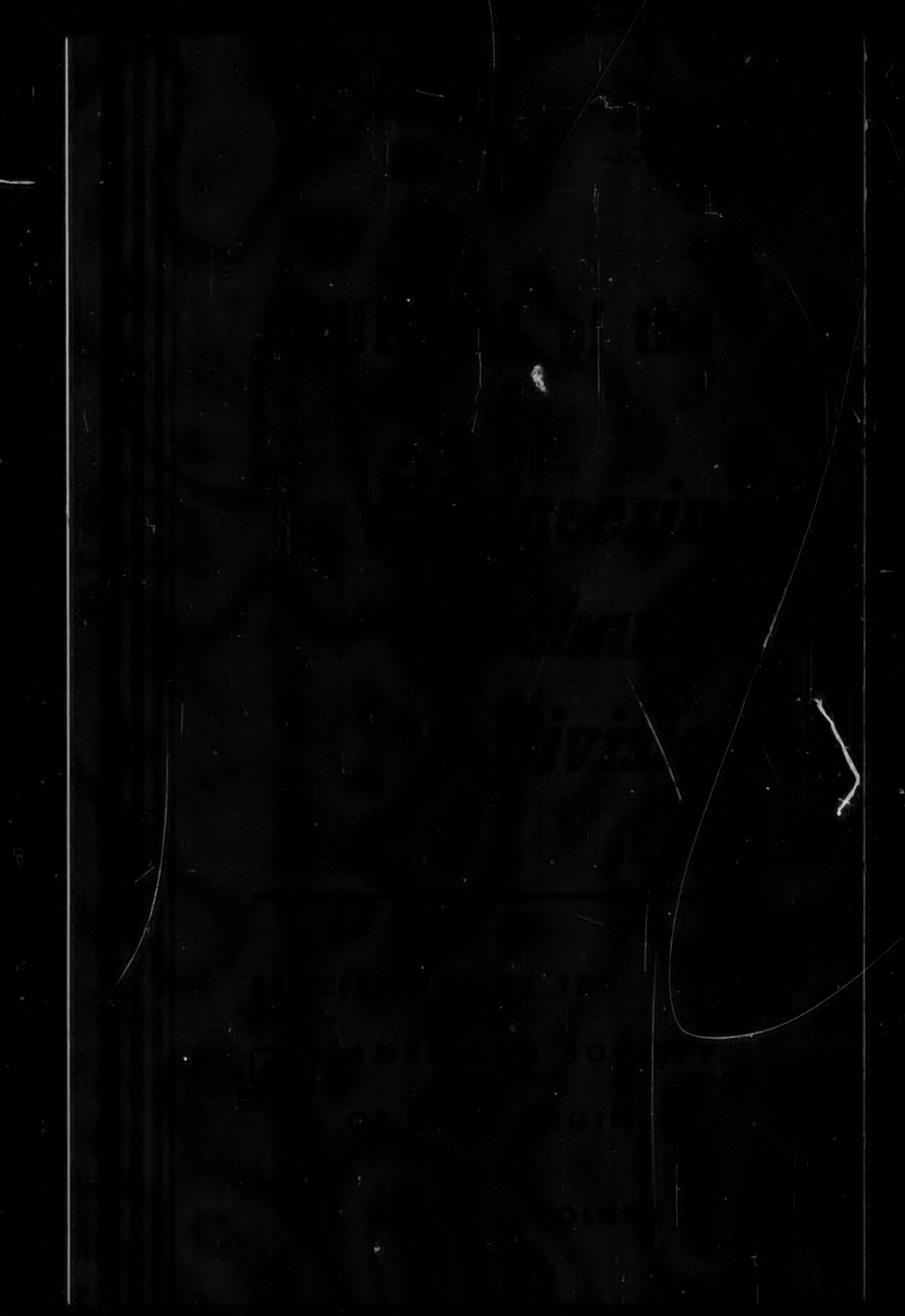

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2. An abstract should precede the paper.

3. Two manuscript (1 ribbon copy and two copies) should be double-spaced on one side of 8½-in. × 11-in. paper. Papers that were originally prepared for oral presentation must be rewritten into the third person before being submitted.

4. The author's full name, membership grade, and present connection should appear on the first page of the paper.

5. Mathematical expressions should be clearly drawn in black ink. Symbols should be carefully identified the first time they appear. Equations should be numbered consecutively with Arabic numerals in parentheses at the right-hand margin of the page. Numbers appearing as superscripts in the text should be in brackets [ ]. Fractions in the text should be written with a solidus (/) rather than horizontally.

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7. Illustrations should be drawn in black ink on one side of 8½-in. × 11-in. paper to form an invisible frame that measures 6½ in. × 10 in. Lettering should also be included within the frame. Glossy photographic prints of the same size should be submitted. Glossy prints not less than 5 in. × 7 in. are acceptable. Explanation and description should be made within the text for each illustration.

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CONTENTS

October, 1956

Papers

Number

Truss Analysis by Stiffness Considerations
by Harold C. Martin 1070

Plastics
by Gordon M. Kline 1071

Plastics: Engineering Materials
by C. H. Adams 1072

Minimum-Weight Design of a Portal Frame
by William Prager 1073

Discussion 1091

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TRUSS ANALYSIS BY STIFFNESS CONSIDERATIONS

Harold C. Martin[1]
(Proc. Paper 1070)

SUMMARY

A general method of analysis, particularly advantageous for redundant structures, is developed in detail for the pin-connected truss. The proposed method enables deflections, member forces, and reactions to be determined for any arbitrary set of applied loads.

It is shown that equilibrium and continuity requirements can be satisfied by following a simple tabular procedure for writing the stiffness matrix of the entire structure. This is the principal task of the analyst and is sufficiently simple to be carried out by non-engineering trained personnel. No extra complications arise from the subsequent introduction of additional redundant members or reactions. Once the table representing the stiffness of the structure has been obtained, the complete solution follows from numerical calculation. This can be carried out using the slide rule, desk calculator, or automatic digital computing equipment.

Changes in design are accounted for by locally correcting the stiffness matrix. Changes in external forces do not affect the analysis.

Any determinate or indeterminate structure may be solved by the suggested method. In this paper the basic procedure is explained and then applied to several examples.

INTRODUCTION

The problem of solving highly redundant structures is important from an engineering standpoint and as a result has received considerable attention in the literature. Since the problem is well known comments on previous work will not be repeated here.

The solution to any structural problem must be based on satisfying certain fundamental conditions. These are well known to the structural analyst and are listed in a following section of this paper. Whereas these basic conditions

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are not subject to change, detailed procedures for applying them to any given problem can vary to a considerable degree. The method presented here will be seen to possess certain advantages which become particularly significant for the highly redundant structure.

In this paper a routine procedure is described for satisfying the fundamental requirements of equilibrium and continuity by essentially developing the stiffness matrix for the structure. This is demonstrated for the truss. Applications to other structures can be made once stiffness expressions for the basic members have been determined.

The main advantage of the method presented here is that it enables a complex truss to be analyzed with a minimum of engineering effort. Numerical operations involving matrices are used to obtain the solution. This is also an advantage, for such calculations are of a routine type and may be effectively carried out on automatic digital computing equipment. A systematized procedure for applying the method is developed in the paper. The structural engineer will recognize that the solution presented here based on matrix computations is merely an alternative to the various types of relaxation methods (as moment distribution) which have been used in the past. However, the recent emergence of truly high speed computers has widened the horizons, so to speak, for the structural engineer, enabling him to undertake analyses which would have been prohibitive a few years ago.

Although not emphasized in the main body of the paper, the proposed method is very useful when structural data is required for a subsequent dynamic analysis. It is then necessary to have available an adequate set of deflection influence coefficients. These are obtained readily from the stiffness matrix developed in this paper.

A great amount of practical detail will suggest itself to the engineer who finds it advantageous to apply this method to structural problems. These are sufficiently numerous to preclude treatment in a paper.

ACKNOWLEDGMENT

This present paper represents an application to pin-connected trusses of a detailed procedure devised at the Boeing Airplane Company for analyzing highly redundant aircraft structures. It was the writer's privilege to be associated in this work with the following men who made notable contributions to the development and application of the method: Mr. M. J. Turner, Structural Dynamics Unit Chief, Boeing Airplane Company, Seattle Division: Prof. R. W. Clough, Department of Civil Engineering, University of California (Berkeley): Mr. J. L. Topp, Structures Engineer, Boeing Airplane Company, Wichita Division. It is expected that the application of the method to shell type structures, as found in aircraft, will be published shortly.

Discussion of Proposed Method

In the analysis of any structure it is generally necessary to satisfy the following conditions:

a) forces must be in equilibrium.
b) deformations must be compatible.
c) forces and displacements must be related according to some prescribed rule.

d) support or boundary conditions must be satisfied.

The various methods of structural analysis represent the different approaches which may be used in satisfying these requirements.

The method presented here applies to the determinate or indeterminate structure. Although applicable in principle to any structure, attention in this paper will be focused on the pin-connected truss.

Prior to consideration of external loads or nature of supports, arbitrary displacements are imposed on each joint (node) of the truss. For example, a given node will be displaced an arbitrary amount in a given direction, while all other components of node displacement are held fixed. The forces required at the displaced node to produce this deflection are recorded, as are the reactions developed at all neighboring nodes (nodes connected directly by a single member to the displaced node). These forces are known from previously determined information based on the stiffnesses of the individual truss members.

Each node is displaced in this manner (successively in x, y, z directions for a space truss) and the corresponding forces and reactions recorded. Superimposing all such results leads to a set of equations relating node displacements to node forces. Since each node has been given an arbitrary displacement in each significant direction, the superposition of all such states will be capable of representing any possible truss deflections which may occur due to application of any set of applied loads. Therefore the force-deflection equations obtained in this manner can be specialized to fit any loaded state for the truss.

For an actual design problem, some nodes will be fixed due to supports provided for the truss. Forces at these nodes then become reactions. All other components of node force can be considered as possible applied loads. By specifying applied loads and treating node displacements as unknown, a solution for these displacements can be obtained out of the set of force-deflection equations. Once these deflections are known, the reactions at the supports can be found. Finally, since all node deflections are now known the forces induced in the individual truss members can be found.

The solution found in this manner will satisfy all the requirements listed previously.

Fortunately the detailed application of the method is essentially tabular in nature. As a result it is a simple matter to set up the initial set of force-deflections equations; in fact, one usually merely sets up the matrix of stiffness influence coefficients relating forces and deflections. Once this matrix has been found the solution for node deflections, external reactions, and separate member forces proceeds from routine matrix operations.

This program is now developed in detail for a single truss member and then applied to several actual problems.

Stiffness - Elastic Spring

In Fig. 1 as elastic spring is deflected an amount δ (inches) by a force F

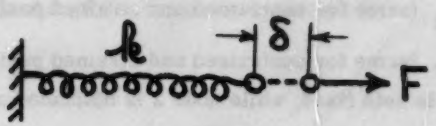

Fig. 1. Elastic Spring

(lbs.). The relation between force and displacement is,

$$F = k\delta \tag{1}$$

where k = spring "stiffness" (lbs./in.). As seen from Eq. 1, k can be interpreted as the force necessary to deflect the spring a unit amount.

From Eq. 1,

$$\delta = k^{-1} F \tag{2}$$

where k^{-1} = spring "flexibility" (in/lb.). It is useful to interpret k^{-1} as the deflection due to a unit load.

Similar expressions relate forces and displacements for complex elastic structures.

Stiffness - Pin-Ended Truss Member

A pin-ended truss member is shown in Fig. 2. Forces can be applied at

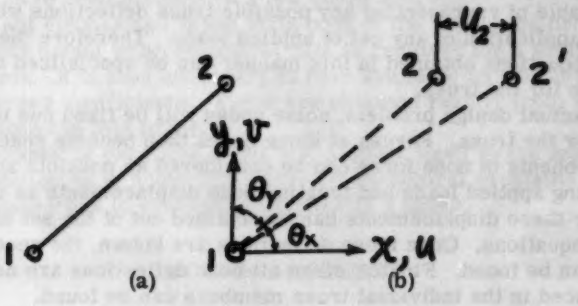

Fig. 2. Pin-ended Truss Member

nodes 1 and 2. In order to write the force-deflection equations the following notation is established:

L = length of unstrained member

A = cross-sectional area of member

E = modulus of elasticity of member

u = displacement in X - direction

v = displacement in y - direction

λ = cos θ_x (same for unstrained and strained position of member)

μ = cos θ_y (same for unstrained and strained position of member)

Suppose node 1 is held fixed, while node 2 is displaced a distance as shown

in Fig. 26. The change in length, L, of the member is then,

$$\Delta L = u_2 \lambda$$

while the axial force P required to produce this change in length is,

$$P = \frac{AE}{L} \Delta L = \frac{AE}{L} u_2 \lambda \tag{3}$$

Components of P at node 2 are,

$$F_{x_2} = P\lambda = \frac{AE}{L} \lambda^2 u_2 \tag{4a}$$

$$F_{y_2} = P\mu = \frac{AE}{L} \lambda\mu u_2 \tag{4b}$$

Equilibrium of forces requires that,

$$F_{x_1} = -F_{x_2} = -\frac{AE}{L} \lambda^2 u_2 \tag{4c}$$

$$F_{y_1} = -F_{y_2} = -\frac{AE}{L} \lambda\mu u_2 \tag{4d}$$

In precisely the same way the member can be subjected to displacement, while all other components of node displacement remain zero. Repeating this for all components of node displacement and superimposing results will lead to the following set of force-deflection equations:

$$\begin{aligned}
F_{x_1} &= \frac{AE}{L}\left(\lambda^2 u_1 + \lambda\mu v_1 - \lambda^2 u_2 - \lambda\mu v_2\right) \\
F_{y_1} &= \frac{AE}{L}\left(\lambda\mu u_1 + \mu^2 v_1 - \lambda\mu u_2 - \mu^2 v_2\right) \\
F_{x_2} &= \frac{AE}{L}\left(-\lambda^2 u_1 - \lambda\mu v_1 + \lambda^2 u_2 + \lambda\mu v_2\right) \\
F_{y_2} &= \frac{AE}{L}\left(-\lambda\mu u_1 - \mu^2 v_1 + \lambda\mu u_2 + \mu^2 v_2\right)
\end{aligned} \tag{5}$$

A convenient way of writing equations (5) is to employ matrix notation. The corresponding matrix form for these equations is,

$$\begin{Bmatrix} F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \tag{6a}$$

or in compact notation,

$$\{F\} = [K]\{\delta\} \tag{6b}$$

where $\{F\}$ is the column matrix of forces, $\{\delta\}$ the column matrix of displacements, and $[K]$ the square stiffness matrix relating forces and displacements.

Equation (6b) corresponds to equation (1) for the elastic spring. The stiffness of the bar is represented by the square matrix $[K]$. It should be noticed that $[K]$ is symmetric; that is, if $k_{\alpha\beta}$ is the element in the α row and β column of $[K]$, then $k_{\alpha\beta} = k_{\beta\alpha}$. This result exhibits the well known reciprocal law for linearly elastic structures.

To discuss the elastic truss member further it is useful to establish supports sufficient to prevent it from moving as a rigid body. This can be accomplished by requiring at least three components of node displacement to be zero. For example suppose $v_1 = u_2 = v_2 = 0$. Then equations (5), or (6) reduce to,

$$F_{x_1} = \frac{AE}{L} \lambda^2 u_1 = -F_{x_2}$$

$$F_{y_1} = \frac{AE}{L} \lambda\mu u_1 = -F_{y_2}$$

Here F_{y1}, F_{x2}, and F_{y2} are reactions, while F_{x1} is the applied load. If F_{x1} is specified, U_1 can be computed. Knowing U_1 the reactions can be found. Finally the truss member axial force P can be found from Eq. (3), with ΔL replaced by $-\lambda U_1$.

The process explained here for the single member can be applied to complex trusses. This will be seen to possess certain definite advantages over the more conventional types of analysis.

Stiffness Analysis - Simple Truss

When the stiffness expressions for individual members of a structure are known, the stiffness of an assemblage of such members may be formed. A simple example is represented by the truss of Fig. 3.

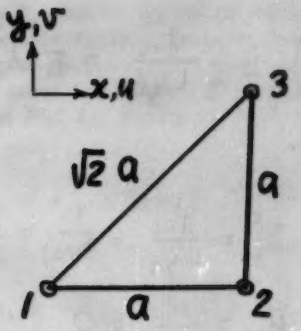

Fig. 3. Simple Truss

For simplicity of explanation it is assumed that all members have equal values of A and E.

The stiffness matrix can be developed by first calculating λ^2, μ^2 and $\lambda\mu$ for each member. This is done in Table I.

TABLE I

Member	x	y	L	$\lambda = \frac{x}{L}$	$\mu = \frac{y}{L}$	λ^2	μ^2	$\lambda\mu$
1-2	a	0	a	1	0	1	0	0
1-3	a	a	$\sqrt{2}$ a	$1/\sqrt{2}$	$1/\sqrt{2}$	1/2	1/2	1/2
2-3	0	a	a	0	1	0	1	0

It is now desired to form the stiffness matrix for the complete truss. This is the principal task which the "engineer" must perform in carrying out the analysis and, as will be seen, it is quite routine in nature.

The stiffness of any single member is given by equation (6a). Since, for a truss, members have various lengths it is advisable to bring this term inside the matrix; that is, every element in the square matrix of equation (a) is to be multiplied by 1/L. (If areas of members are different A should also be brought inside the matrix.)

Since for the truss members,

$$L_{1\text{-}2} = a \quad , \quad L_{1\text{-}3} = \sqrt{2}\, a \quad , \quad L_{2\text{-}3} = a$$

it is convenient to keep AE/a outside the truss stiffness matrix. Individual member stiffness can then be written in terms of $\bar{\lambda}^2$, $\bar{u}^2$, and $\bar{\lambda}\bar{u}$ with a common factor AE/a where,

$$\left(\frac{\lambda^2}{L}\right)_{i-j} = \frac{1}{a}\frac{\lambda_{i-j}^2}{L_{i-j}/a} = \frac{1}{a}\bar{\lambda}_{i-j}^2$$

For member 1-2,

$$\bar{\lambda}_{1-2}^2 = \frac{\lambda_{1-2}^2}{L_{1-2}/a} = \frac{1}{a/a} = 1$$

$$\bar{\mu}_{1-2}^2 = \frac{\mu_{1-2}^2}{L_{1-2}/a} = 0$$

$$(\bar{\lambda}\bar{\mu})_{1-2} = \frac{(\lambda\mu)_{1-2}}{L_{1-2}/a} = 0$$

etc. for the other members. Table II presents these values.

TABLE II

Member	$\bar{\lambda}^2$	$\bar{\mu}^2$	$\bar{\lambda}\bar{\mu}$
1-2	1	0	0
1-3	$1/2\sqrt{2}$	$1/2\sqrt{2}$	$1/2\sqrt{2}$
2-3	0	1	0

Equation (6a) for the complete truss can now be written directly from Table II. The result is given first and then explained in some detail.

$$\begin{Bmatrix} F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \end{Bmatrix} = \frac{AE}{a} \begin{bmatrix} 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -1 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & -1 & \frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad (8)$$

Consider the first column in the above square stiffness matrix. The first

element represents force F_{x1} due to u_1, the second element force F_{y1} due to u_1, etc. Similarly the second column represents these forces due to displacement v_1, (all other node displacements = 0), etc. for the other columns.

Imposing u_1, while all other node displacements remain zero, strains members 1-2 and 1-3 (see Fig. 3). Noting the form of equation (6b) it is then seen that for the truss,

$$F_{x_1} = \bar{\lambda}^2_{1-2} + \bar{\lambda}^2_{1-3} = 1 + 1/2\sqrt{2}$$

$$F_{y_1} = (\bar{\lambda}\bar{\mu})_{1-2} + (\bar{\lambda}\bar{\mu})_{1-3} = 1/2\sqrt{2}$$

$$F_{x_2} = -\bar{\lambda}^2_{1-2} = -1 \qquad \text{etc.}$$

Following the form of equation (6b) and using values in Table II all the columns in equation (8) can be written in turn.

Two checks can be applied to the stiffness matrix. First it must be symmetric in the sense previously defined for the single truss member. Second, for each column the sum of X - forces must vanish and likewise for y - forces.

The square matrix of equation (8) is the stiffness matrix for the truss. Solution for displacements, reactions, and member forces now follow from a sequence of matrix operations. These are numerical in nature and can be carried out on a desk calculator or on automatic digital calculating equipment.

Before proceeding with the solution supports must be specified for the truss. These can be chosen in any desired way, it is only necessary that rigid body motion be prevented. For the present problem nodes 2 and 3 will be fixed, while node 1 is kept free.
Equation (8) is then written as;

$$\begin{Bmatrix} F_{x_1} \\ F_{y_1} \\ \hline F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \end{Bmatrix} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{12}' & K_{22} \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ \hline u_2=0 \\ v_2=0 \\ u_3=0 \\ v_3=0 \end{Bmatrix} \qquad (9)$$

where K_{11} --- K_{22} are obtained from K of equation (8),

$$[K_{11}] = \frac{AE}{a}\begin{bmatrix} 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}, \qquad [K_{12}] = \frac{AE}{a}\begin{bmatrix} -1 & 0 & -1/2\sqrt{2} & -1/2\sqrt{2} \\ 0 & 0 & -1/2\sqrt{2} & -1/2\sqrt{2} \end{bmatrix}$$

$$[K_{12}]' = \frac{AE}{a}\begin{bmatrix} -1 & 0 \\ 0 & 0 \\ -1/2\sqrt{2} & -1/2\sqrt{2} \\ -1/2\sqrt{2} & -1/2\sqrt{2} \end{bmatrix} = \text{transpose of } [K_{12}]$$

$$[K_{12}] = \frac{AE}{a}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1/2\sqrt{2} & 1/2\sqrt{2} \\ 0 & -1 & 1/2\sqrt{2} & 1+1/2\sqrt{2} \end{bmatrix}$$

These are seen to be submatrices of K in equation (8).

In equation (9) F_{x_1} and F_{x_2} are applied loads, while the other forces are the unknown external reaction. Also u_1 and u_2 are the unknown displacements. Solution for the unknown quantities comes by expanding equation (9) into the two sets of equations (by matrix multiplication),

$$\begin{Bmatrix} F_{x_1} \\ F_{y_1} \end{Bmatrix} = [K_{11}] \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} \tag{10a}$$

$$\begin{Bmatrix} F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \end{Bmatrix} = [K_{12}]' \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} \tag{10b}$$

From the first equation above,

$$\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = [K_{11}]^{-1} \begin{Bmatrix} F_{x_1} \\ F_{y_1} \end{Bmatrix} \tag{11a}$$

which gives displacements in terms of any set of applied loads. Substituting (11a) into (10b),

$$\begin{Bmatrix} F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \end{Bmatrix} = [K_{12}]' [K_{11}]^{-1} \begin{Bmatrix} F_{x_1} \\ F_{y_1} \end{Bmatrix} \tag{11b}$$

which gives unknown reactions in terms of any set of applied forces.

The matrix operations indicated in equations (11a) and (11b) are standard

operations. Equation (11a) corresponds to equation (2); as such $[K_{11}]^{-1}$ represents the deflection or flexibility influence coefficients for the given truss. Note that $[K_{11}]$ can be obtained from $[K]$ of equation (8) by striking out columns and corresponding rows for which zero displacements (due to supports) have been specified.

The final step is that of determining truss member forces. By generalizing equation (3) it can be shown that the force in member i-j due to displacements u and v of its nodes (at i and j) is given by,

$$P_{i-j} = \left(\frac{AE}{L}\right)_{i-j} \left[\lambda_{i-j}\left(u_j - u_i\right) + \mu_{i-j}\left(v_j - v_i\right)\right] \tag{12}$$

Such a relation holds for each member of the truss. Displacements are known in terms of applied loads from equation (11a). Therefore, the member forces can be calculated from the above equation. A matrix procedure can be used.

For the problem at hand the following results are obtained:

a) displacements

$$\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \frac{a}{AE} \begin{bmatrix} 1 & -1 \\ -1 & 1+2\sqrt{2} \end{bmatrix} \begin{Bmatrix} F_{x_1} \\ F_{y_1} \end{Bmatrix} \tag{13a}$$

b) reactions

$$\begin{Bmatrix} F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \end{Bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} F_{x_1} \\ F_{y_1} \end{Bmatrix} \tag{13b}$$

c) member forces

$$\begin{Bmatrix} P_{1-2} \\ P_{1-3} \end{Bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -\sqrt{2} \end{bmatrix} \begin{Bmatrix} F_{x_1} \\ F_{y_1} \end{Bmatrix} \tag{13b}$$

These can be readily checked against a conventional analysis for this simple structure.

Comments

Although the procedure just described may seem strange and a bit awkward at first, it does possess some real advantages for the structural engineer.

These are as follows:

a) The procedure for setting up the stiffness matrix is entirely routine and is not complicated by the presence of redundant members or redundant reactions.

b) Once the stiffness matrix has been found the solution for deflections, reactions, and member forces follows from strictly numerical matrix operations. These can be efficiently carried out using a desk calculator or automatic digital computing equipment.

c) Any set of applied loads can be inserted as the last step in the calculations.

d) Changes in design - internal members or reactions - only affect local portions of the stiffness matrix and subsequent numerical calculations. As a result several design configurations can be investigated without undue engineering effort.

e) The method applies equally well to determinate or undeterminate structures.

f) Structures containing different kinds of load carrying members - axial force members, beams, torque cells, and so on - and possessing a high order of redundancy are especially suited to this method.

Analysis - Indeterminate Truss

As a more representative problem the truss of Fig. 4 will now be analyzed by the preceding method.

Nodes are numbered 1 through 8. Lengths or members can be determined from dimensions on sketch. As before A and E will be assumed the same for each member. Supports are established which fix nodes 1, 2, 7, and 8. The truss therefore has three internal redundant members and five redundant components of external reactions.

It is desired to find u and v components of displacement for nodes 3, 4, 5, and 6 due to any loading applied at these same nodes. In addition all internal member forces and all reactions are required.

As for the simpler problem of Fig. 3, a table is first set up giving λ^2, . . . etc. for each member. Conversion to $\overline{\lambda}^2$, . . etc. is made in this same table; the common factor on all truss member stiffnesses is AE/10.

The stiffness matrix is now developed precisely as for the simple truss, previously discussed. In doing this $\overline{\lambda}^2$, . . etc. of Table III are used and the common factor outside the matrix will be AE/10. The order of the original stiffness matrix will be 16 x 16. This is shown in Table IV.

For the truss of Fig. 4 zero displacements are shown at nodes 1, 2, 7, and 8. It is therefore convenient to rearrange the matrix of Table IV to agree with the following form for the corresponding force-deflection equations:

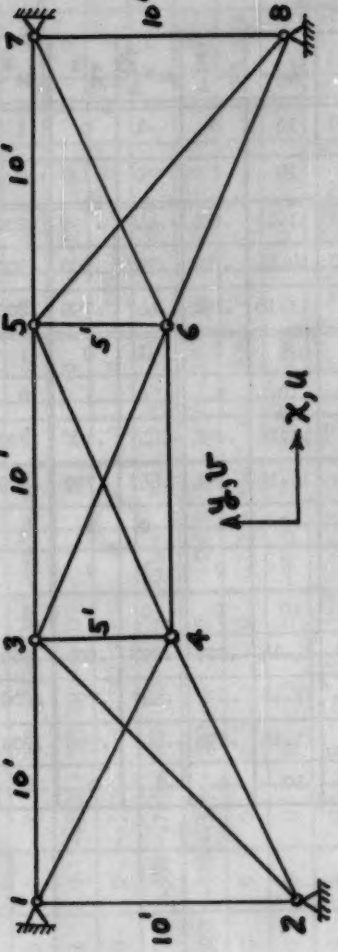

Fig. 4. Indeterminate Truss

TABLE III

Member	x	y	L	$\lambda=\frac{x}{L}$	$\mu=\frac{y}{L}$	λ^2	μ^2	$\lambda\mu$	$\bar{\lambda}^2$	$\bar{\mu}^2$	$\bar{\lambda}\bar{\mu}$
1-2	0	-10	10	0	-1	0	1	0	0	1	0
1-3	10	0	10	1	0	1	0	0	1	0	0
1-4	10	-5	11.18	.894	-.447	.799	.200	-.400	.715	.179	-.358
2-3	10	10	14.14	.707	.707	.500	.500	.500	.353	.353	.353
2-4	10	5	11.18	.894	.447	.799	.200	.400	.715	.179	.358
3-4	0	5	5	0	-1	0	1	0	0	2	0
3-5	10	0	10	1	0	1	0	0	1	0	0
3-6	10	-5	11.18	.894	-.447	.799	.200	-.400	.715	.179	-.358
4-5	10	5	11.18	.894	.447	.799	.200	.400	.715	.179	.358
4-6	10	0	10	1	0	1	0	0	1	0	0
5-6	0	5	5	0	-1	0	1	0	0	2	0
5-7	10	0	10	1	0	1	0	0	1	0	0
5-8	10	-10	14.14	.707	-.707	.500	.500	-.500	.353	.353	-.353
6-7	10	5	11.18	.894	.447	.799	.200	.400	.715	.179	.358
6-8	10	-5	11.18	.894	-.447	.799	.200	-.400	.715	.179	-.358
7-8	0	10	10	0	-1	0	1	0	0	1	0

TABLE IV

Elements in Indeterminate Truss Stiffness Matrix

(Common factor multiplying all elements = AE/10

	u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5	u_6	v_6	u_7	v_7	u_8	v_8
F_{x_1}	1.715	-.358	0	0	-1	0	-.715	.358	0	0	0	0	0	0	0	0
F_{y_1}	-.358	1.179	0	-1	0	0	.358	-.179	0	0	0	0	0	0	0	0
F_{x_2}	0	0	1.068	.711	-.353	-.353	-.715	-.358	0	0	0	0	0	0	0	0
F_{y_2}	0	-1	.711	1.532	-.353	-.353	-.358	-.179	0	0	0	0	0	0	0	0
F_{x_3}	-1	0	-.353	-.353	3.068	-.005	0	0	-1	0	-.715	.358	0	0	0	0
F_{y_3}	0	0	-.353	-.353	-.005	2.532	0	-2	0	0	.358	-.179	0	0	0	0
F_{x_4}	-.715	.358	-.715	-.358	0	0	3.145	.358	-.715	-.358	-1	0	0	0	0	0
F_{y_4}	.358	-.179	-.358	-.179	0	-2	.358	2.537	-.358	-.179	0	0	0	0	0	0
F_{x_5}	0	0	0	0	-1	0	-.715	-.358	3.068	.005	0	0	-1	0	-.353	.353
F_{y_5}	0	0	0	0	0	0	-.358	-.179	.005	2.532	0	-2	0	0	.353	-.353
F_{x_6}	0	0	0	0	-.715	.358	-1	0	0	0	3.145	-.358	-.715	-.358	-.715	.358
F_{x_7}	0	0	0	0	.358	-.179	0	0	0	-2	-.358	2.537	-.358	-.179	.358	-.179
F_{x_7}	0	0	0	0	0	0	0	0	-1	0	-.715	-.358	1.715	.358	0	0
F_{y_7}	0	0	0	0	0	0	0	0	0	0	-.358	-.179	.358	1.179	0	-1
F_{x_8}	0	0	0	0	0	0	0	0	-.353	.353	-.715	.358	0	0	1.068	-.711
F_{y_8}	0	0	0	0	0	0	0	0	.353	-.353	.358	-.179	0	-1	-.711	1.532

$$\left\{\begin{array}{c} F_{x_3} \\ F_{y_3} \\ F_{x_4} \\ F_{y_4} \\ F_{x_5} \\ F_{y_5} \\ F_{x_6} \\ F_{y_6} \\ \hline F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \\ F_{x_7} \\ F_{y_7} \\ F_{x_8} \\ F_{y_8} \end{array}\right\} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{12}' & K_{22} \end{array}\right] \left\{\begin{array}{c} u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \\ \hline u_1=0 \\ v_1=0 \\ u_2=0 \\ v_2=0 \\ u_7=0 \\ v_7=0 \\ u_8=0 \\ v_8=0 \end{array}\right\} \qquad (14a)$$

or in compact form,

$$\left\{\begin{array}{c} F_{3,4,5,6} \\ \hline F_{1,2,7,8} \end{array}\right\} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{12}' & K_{22} \end{array}\right] \left\{\begin{array}{c} \delta_{3,4,5,6} \\ \hline 0 \end{array}\right\} \qquad (14b)$$

Here $F_{3,4,5,6}$ represent any possible set of applied loads and $F_{1,2,7,8}$ the unknown external reactions. Also $\delta_{3,4,5,6}$ represent all unknown node deflextions of the truss. The submatrices K_{11}, . . etc. are listed below for the reader who wishes to check the problem in detail.

$$\left[K_{11}\right] = \frac{AE}{10} \begin{bmatrix} 3.068 & -.005 & 0 & 0 & -1 & 0 & -.715 & .358 \\ -.005 & 2.532 & 0 & -2 & 0 & 0 & .358 & -.179 \\ 0 & 0 & 3.145 & .358 & -.715 & -.358 & -1 & 0 \\ 0 & -2 & .358 & 2.537 & -.358 & -.179 & 0 & 0 \\ -1 & 0 & -.715 & -.358 & 3.068 & .005 & 0 & 0 \\ 0 & 0 & -.358 & -.179 & .005 & 2.532 & 0 & -2 \\ -.715 & .358 & -1 & 0 & 0 & 0 & 3.145 & -.358 \\ .358 & -.179 & 0 & 0 & 0 & -2 & -.358 & 2.537 \end{bmatrix}$$

$$\left[K_{12}\right] = \frac{AE}{10}$$

$$\begin{bmatrix}
-1 & 0 & -.353 & -.353 & 0 & 0 & 0 & 0 \\
0 & 0 & -.353 & -.353 & 0 & 0 & 0 & 0 \\
-.715 & .358 & -.715 & -.358 & 0 & 0 & 0 & 0 \\
.358 & -.179 & -.358 & -.179 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -.353 & .353 \\
0 & 0 & 0 & 0 & 0 & 0 & .353 & -.353 \\
0 & 0 & 0 & 0 & -.715 & -.358 & -.715 & .358 \\
0 & 0 & 0 & 0 & -.358 & -.179 & .358 & -.179
\end{bmatrix}$$

$$\left[K_{22}\right] = \frac{AE}{10}$$

$$\begin{bmatrix}
1.715 & -.358 & 0 & 0 & 0 & 0 & 0 & 0 \\
-.358 & 1.179 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.068 & .711 & 0 & 0 & 0 & 0 \\
0 & -1 & .711 & 1.532 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.715 & .358 & 0 & 0 \\
0 & 0 & 0 & 0 & .358 & 1.179 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.068 & -.711 \\
0 & 0 & 0 & 0 & 0 & -1 & -.711 & 1.532
\end{bmatrix}$$

The solution proceeds now exactly as it did for the simple truss. From equation (14b)

$$\left\{F_{3,4,5,6}\right\} = \left[K_{11}\right]\left\{\delta_{3,4,5,6}\right\} \qquad (15a)$$

and,

$$\left\{F_{1,2,7,8}\right\} = \left[K_{12}\right]'\left\{\delta_{3,4,5,6}\right\} \qquad (15b)$$

From equation (15a),

$$\left\{\delta_{3,4,5,6}\right\} = \left[K_{11}\right]^{-1}\left\{F_{3,4,5,6}\right\} \qquad (16a)$$

and substituting this into equation (15b)

$$\{F_{1,2,7,8}\} = [K_{12}]' [K_{11}]^{-1} \{F_{3,4,5,6}\} \quad (16b)$$

The laborious part of the above analysis is that of finding $[K_{11}]^{-1}$. Since the matrix involved is 8 x 8 this is a sizable calculation. The result given below was obtained, however, in less than five minutes by utilizing a high speed digital calculating machine. The numerical results are: (see page 19). Used in equations (16a) and (16b) these results then give all components of node deflection and all components of external reactions due to any specified set of external forces (applied at nodes 3, 4, 5, and 6).

Truss member forces may be calculated by applying equation (12) to each member of the truss. Results are, (see page 20).

$$\left[K_{11}\right]^{-1} = \frac{10}{AE}$$

$$\begin{bmatrix}
.410269 & -.044793 & .060461 & -.029113 & .144549 & -.080315 & .105128 & -.109534 \\
-.044793 & 1.267086 & -.115404 & 1.048185 & .080315 & .307294 & -.155132 & .316080 \\
.060461 & -.115404 & .430691 & -.125972 & .105128 & .155132 & .178721 & .130841 \\
-.029113 & 1.048185 & -.125972 & 1.275019 & .109534 & .316080 & -.130841 & .308777 \\
.144549 & .080315 & .105128 & .109534 & .410269 & .044793 & .060461 & .029113 \\
-.080315 & .307294 & .155132 & .316080 & .044793 & 1.267086 & .115404 & 1.048185 \\
.105128 & -.155132 & .178721 & -.130841 & .060461 & .115404 & .430691 & .125972 \\
-.109534 & .316080 & .130841 & .308777 & .029113 & 1.048185 & .125972 & 1.276019
\end{bmatrix}$$

$$\left[K_{12}\right]'\left[K_{11}\right]^{-1} =$$

$$\begin{bmatrix}
-.463921 & .502557 & .413503 & .575998 & -.180502 & .082552 & -.279754 & .126525 \\
.026856 & -.228940 & .176736 & -.273505 & .018029 & -.001041 & .087403 & -.008430 \\
-.16820 & -.724205 & -.243451 & -.726477 & -.193757 & -.304200 & -.063292 & -.277004 \\
-.145447 & -.577780 & -.112243 & -.543042 & -.136619 & -.192240 & -.022910 & -.175023 \\
-.180502 & -.082552 & -.279754 & -.126525 & 0.463921 & -.502557 & -.413503 & -.575998 \\
-.018029 & -.001041 & -.087430 & -.008430 & -.026856 & -.228940 & -.176736 & -.273505 \\
-.193757 & .304200 & .277004 & .277004 & -.161820 & .724205 & -.243451 & .726477 \\
.136619 & -.192240 & -.175023 & -.175023 & .145447 & -.577780 & .112243 & -.543042
\end{bmatrix}$$

$$
\begin{Bmatrix} P_{1-3} \\ P_{2-3} \\ P_{1-4} \\ P_{2-4} \\ P_{3-4} \\ P_{3-5} \\ P_{4-5} \\ P_{3-6} \\ P_{4-6} \\ P_{5-6} \\ P_{5-7} \\ P_{6-7} \\ P_{5-8} \\ P_{6-8} \end{Bmatrix} =
\begin{bmatrix}
.41027 & -.04479 & .06046 & -.02911 & .11455 & -.08032 & .10513 & -.10953 \\
.18274 & .61115 & -.02747 & .50954 & .11244 & .11349 & -.02501 & .10328 \\
.06001 & -.51160 & .39494 & -.61119 & .04029 & -.11233 & .19532 & -.01884 \\
.03673 & -.32696 & .29416 & .40963 & .12791 & .25053 & .09064 & .22818 \\
-.03136 & .43780 & .02114 & -.45566 & -.05844 & -.01758 & -.04858 & .01460 \\
-.26572 & .12511 & .04467 & .13865 & .26572 & .12511 & -.04467 & .13864 \\
.04679 & -.13978 & -.11801 & -.19558 & .21821 & .29213 & .00389 & .21438 \\
-.21821 & .29213 & -.00389 & .21438 & -.04679 & -.21378 & .11801 & -.19558 \\
.04467 & -.03973 & -.25197 & -.00487 & -.04467 & -.03973 & .25197 & -.00487 \\
.05844 & -.01758 & .04858 & .01460 & .03136 & .43780 & -.02114 & -.45566 \\
-.11455 & -.08032 & -.10513 & -.10953 & -.11027 & -.04479 & -.06046 & -.02911 \\
-.04029 & -.00233 & -.19532 & -.01884 & -.06001 & -.51160 & -.39494 & -.61119 \\
-.03212 & .11349 & .02501 & .10328 & -.18274 & .61145 & .02747 & .50953 \\
-.12791 & .25053 & & .22808 & -.03673 & .32696 & -.29416 & .40963
\end{bmatrix}
\begin{Bmatrix} F_{13} \\ F_{23} \\ F_{24} \\ F_{34} \\ F_{35} \\ F_{45} \\ F_{36} \\ F_{46} \end{Bmatrix}
$$

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PLASTICS [a]

Gordon M. Kline[1]
(Proc. Paper 1071)

The plastics industry emerged from Word War II with new materials and techniques destined to play an important role in the growth of the American mass production industries. These developments spearheaded a 4-fold increase in the production of plastics during the past 10 years,—from approximately 900 million pounds in 1945 to 3.5 billion pounds in 1955. The variety and versatility of the plastic materials now available and the applications that these materials are finding in product engineering are considered in this evaluation of their present and future roles as basic materials of construction.

It is important to remember that when we are talking of plastics, we are talking of many things. The multisyllable names of some of these plastics, such as acrylic, melamine, tetrafluoroethylene, and zein, may appear baffling to the engineer, although he has become accustomed to hearing and speaking of aluminum, magnesium, titanium, and zirconium. These plastic materials differ among themselves as markedly as metal does from glass and rubber from wood. It is reasonable to expect that the chemical names of the common plastics will be as familiar to the next generation as the names of the common metals and alloys are to this one.

The plastics industry in this country originated with the work of Hyatt on celluloid in 1869. It received its greatest stimulus with the introduction of the first commercial synthetic resen (phenolic type) by Baekeland in 1909. In 1925 these two materials were still the only principal types represented in the 28 million pounds produced that year. Production exceeded the 100 million pound mark for the first time ten years later, 1935. Thus, the bulk of the growth of the plastics industry, both in volume of production (Fig. 1) and in diversity of materials (as will be shown later), has taken place during the past 20 years. Production statistics for the leading commercial plastics in 1955 are presented in Table I.

The growing impact of plastics on practically every segment of industry was highlighted in a survey of the nation's resources and production prepared

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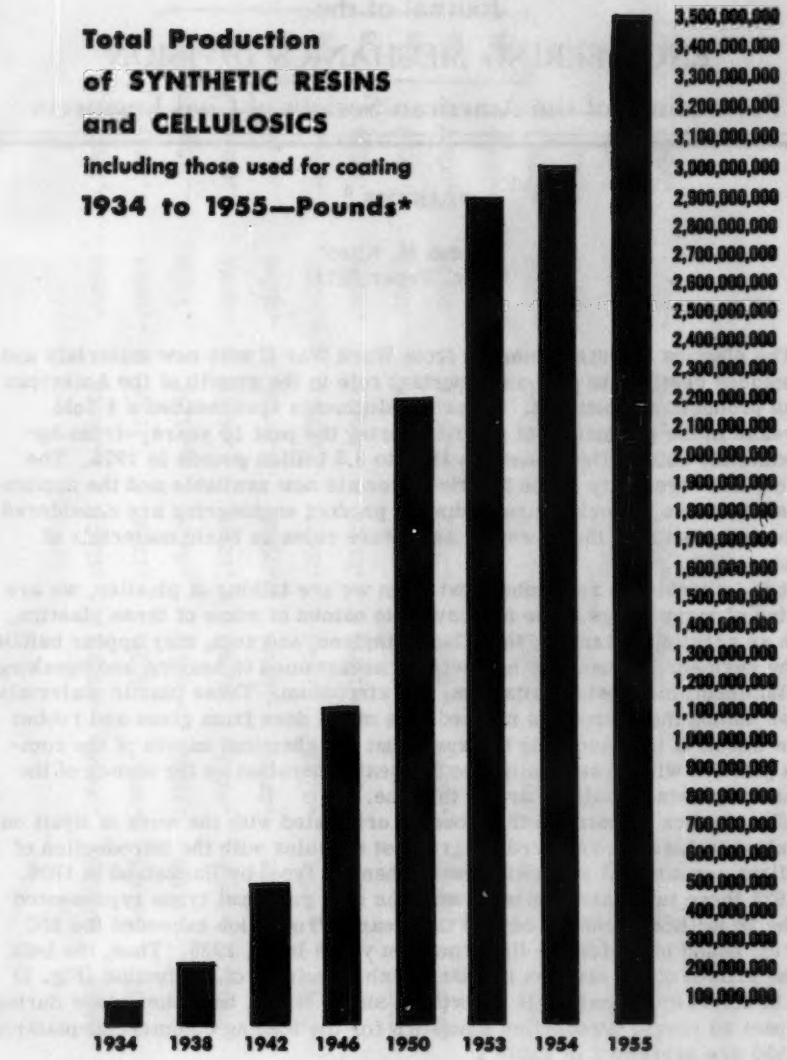
a. Presented at a meeting of the ASCE, Dallas, Tex., Feb. 17, 1956.

1. National Bureau of Standards, U. S. Dept. of Commerce, Wash. D. C.



* Source: U.S. Tariff Commission and MODERN PLASTICS estimates.

FIG. 1

Table I - 1955 Production of Plastics[a]

Material	10^6 lb.
Phenolics	506
Ureas and Melamines	313
Polyesters	55
Vinyls	702
Styrenes	610
Polyethylenes	420
Cellulosics	145
Alkyds and Rosin Esters	567
Coumarones	260
Miscellaneous	137
Total	3,715

[a] Source: U. S. Tariff Commission.

Table II - End Use Consumption of Phenolics in 1955[a]

Use	Percentage
Molding materials	44
Laminating resins	15
Binder for rock wool and fibrous glass insulation	9
Plywood Manufacture	8
Protective coatings	5
Adhesives	5
Brake linings	4
Abrasive wheels	3
Miscellaneous	7
Total	100

[a] Source: Modern Plastics (January 1956 issue).

by the Paley Commission(1). This group reported that "various industry estimates for the growth of a plastics production indicate a likely figure of about 22 billion lb. by 1975 or 10 times current production. This is fairly close to an estimate of the Stanford Research Institute of 25 billion lb. for 1975 (2) and checks an estimate made by a technical group from the Koppers Co. (1, vol. IV), but it is much higher than that of Egloff (1, vol. IV)." The latter's survey of the consumption of petrochemicals includes a forecast of a total production of 3.5 billion lb. of plastics in the United States in 1955, 4.8 billion in 1960, and 9 billion in 1975. The Commission observed that plastics have in the past and will in the future substantially supplement the supply of the following materials by replacing them in various applications: copper, zinc, lead, tin, steel, glass, leather, ceramics, rubber, and wood.

Raw Materials for Plastics

The first step in the manufacture of plastics is the synthesis of a large group of simple chemicals, such as ammonia, acetylene, and hydrochloric acid. These chemicals are then used to prepare the organic compounds, still very small in size, that are the building blocks for the reactive resin-forming substances. These latter substances are called "monomers"; the resin formed by the combination of a large number of molecules of a given monomer is known as a "polymer."

Thus it can readily be seen that basically the plastics industry is a branch of the chemical industry. This accounts for the fact that many of the well-known chemical firms in the United States are associated with the manufacture of plastics. It also explains why so many of the names of common plastics are "jawbreakers".

One early simplified version of the origin of plastics had them derived from "coal, air, lime, and water." If we add four more basic natural substances—petroleum, cellulose, sulfur and salt—we can account for just about all of the commercial types of plastics.

Coal

Because in the early days of the industry coal was the main source of organic chemicals, it is generally given top billing in the genesis of plastics. The coke ovens still yield the important by-products—phenol, cresol, and xylenol—for the phenolic type resins. Another valuable by-product is naphthalene which is oxidized to phthalic anhydride, used in large quantities for the manufacture of alkyd resins and plasticizers. Coumarone and indene, bases for industrially important resins, are also obtained from coal-tar distillate. Benzene, from which synthetic phenol and styrene are made, is available in large quantities by the destructive distillation of coal. In addition to these important products obtained directly from coal, its use in the form of coke for the manufacture of calcium carbide and water gas serves to enhance its position among the raw materials for plastics.

Petroleum

During recent years the petroleum industry, together with the natural gas sources of hydrocarbons, has become an even more prolific source of materials for plastics than coal. Cracking of petroleum by pyrolysis yields

the hydrocarbon gases—ethylene, propylene, and butylenes. Ethylene is used to make vinyl chloride and glycols. Propylene yields allyl alcohol and glycerol. Butylenes can be converted into butadiene and the versatile butyl alcohols. Synthetic benzene is formed by the polymerization of the cracked products of petroleum or natural gas. This in turn can be converted into synthetic phenol for phenolic resin production and into adipic acid and hexamethylenediamine for nylon resin manufacture. Styrene, a monomer needed for the production of polystyrene and synthetic rubber, is made from two products obtainable from petroleum—benzene and ethylene.

Cellulose

Wood and cotton are the primary sources of cellulose for the plastics industry. It is one raw material which can be regenerated as an annual crop, as desired. Cellulose reacts with various chemicals to form esters, ethers, and other derivatives that are useful as bases for plastics.

We have seen how some of the organic chemicals needed as building blocks for plastics are prepared from common industrial raw materials. The next step is the combination of these building blocks to form the prefabricated units (monomers), which are finally joined together to form the resinous products (polymers). This is the point at which matters become complicated by the confusing variety and unlimited possibilities of the combinations. Fortunately, the 100 or so plastics now available can be classified into a relatively few types.

Classes of Plastics

Three categories of plastics are recognized for the sake of convenience in discussing their preparation and properties: 1) thermosetting resins; 2) thermoplastic resins; 3) cellulose derivatives. Thermosetting materials are those that harden when heat is applied. Thermosetting resins have to be formed relatively quickly in order to attain the desired shape before they become rigid, due to the chemical reaction that takes place when they are heated. Thermoplastic materials are those that can be repeatedly softened by heat and hardened by cooling. No chemical reaction is involved in the shaping of thermoplastic resins. In general, cellulose plastics are thermoplastic in their behavior toward heat.

Thermosetting Resins

There are seven types of thermosetting resins that are of major industrial significance: 1) phenolic; 2) urea; 3) melamine; 4) silicon; 5) unsaturated polyester; 6) epoxy; and 7) isocyanate.

Phenolic Resins

The first and still the most versatile of the commercial synthetic resins, the phenolformaldehyde condensation product, was described and patented in 1909 by Dr. Baekeland. The phenolic resins lead the thermosetting resin field in volume of production, approximately 460 million pounds being made in 1955. They are used in molding compounds, laminates, casting compounds, adhesives, coatings, impregnants, and foamed products. A wide range of

physical properties may be obtained by proper choice of resin, filler, and reinforcing agent. The manufacture of articles from phenolic molding powders or fabrication of laminated phenolic products requires high temperatures and pressures.

Phenolics are opaque and generally are produced in black and brown colors. Moldings are hard and rigid; special impact-resistant grades are available. Ordinary grades are resistant to heat up to 300° F. They are poor conductors of heat which makes them useful when used for handles of heated containers. They are excellent electrical insulators. Phenolics are resistant to chemicals and retain their luster, strength, and rigidity under humid conditions.

Phenolic plastics are used in the manufacture of electrical parts, housings for electrical, x-ray, therapeutic and various scientific instruments, and for trays, handles, knobs, containers, protective headgear, gears, pulleys, grinding wheels, radio equipment and protective coatings. Since phenolic plastics are not softened by heat, they can be used advantageously in articles that require cleansing in hot solutions, sterilizing, and resistance to moderately high temperatures of phenolics. Leading end uses in 1955 are shown in Table II.

Urea Resins

The introduction of the urea-formaldehyde molding compounds on the American market in 1929 brought unlimited color possibilities, including white and pastel shades, into the field of thermosetting plastics. It also provided a room-temperature-setting resinous adhesive which soon found extensive use in the manufacture of plywood. Resistance to continuous heat is limited to about 170° F. The urea resins are used in the manufacture of tableware, trays, containers, translucent light shades, instrument housings, protective coatings, and textile finishes.

Melamine Resins

Another amino plastic, a melamine-formaldehyde condensation product, became available in 1939. Melamine plastics are similar in appearance to urea plastics, but are superior in resistance to water, heat and chemicals, scratch hardness, and electrical insulating characteristics. The initial large-scale use of the melamine plastic was in a mineral-filled molding compound for making ignition parts of aircraft engines because of its superior anti-tracking properties in high-altitude flying. Other melamine molding compounds have proved to be eminently satisfactory for tableware and serving trays for ship, aircraft, and hospital use. More recently, melamine-formaldehyde resin has been mixed with pulp fibers to provide a map paper of high wet strength, suitable for handling under adverse field conditions. Panel boards made by bonding glass fabric with this resin have been developed to provide Navy ships with a fire- and arc-resistant laminate of high impact strength. Decorative sheets that can be applied to laminates at pressures as low as 250 p.s.i. have been prepared with melamine resin. Leading end uses of urea and melamine resins in 1955 are shown in Table III.

Table III - End Use Consumption of Urea and Melamine Resins in 1955[a]

Use	Percentage
Plywood manufacture	31
Molding materials, etc.	27
Textile treatments	14
Adhesives	10
Protective coatings	10
Paper treatments	8
Total	100

[a] Sources: U. S. Tariff Commission reports and Modern Plastics (January 1956 issue).

Table IV - End Use Consumption of Vinyl Chloride Plastics in 1955[a]

Use		Percentage
Molding and extrusion		36
	Wire insulation	16
	Phonograph records	5
	Plastisol moldings	5
	Belts and gaskets	4
	Garden hose	3
	Miscellaneous	3
Film and sheeting		26
Fabric and paper treatment		13
Flooring		11
Protective coatings		5
Miscellaneous		9
Total		100

[a] Sources: U. S. Tariff Commission reports and Modern Plastics (January 1956 issue).

Silicon Resins

The organosilicon resins are composed of a network of silicon and oxygen atoms with hydrocarbon radicals attached to the silicon. This silicon-oxygen-silicon structure in high polymers was previously known only in inorganic products such as quartz, glass, asbestos, and mica. Silicone resins were first made available commercially in 1943. The silicone resins found immediate applications in the electrical field because of their resistance to heat, up to 500° F. In combination with glass-fiber tapes and fabrics, they represent a revolutionary advance in electrical insulation. Silicone rubber has found use as a gasket material on searchlights and aircraft. Baked coatings based on silicone resins have been developed for protection of ranges, radiators, heat exhaust pipes, and stacks. Treatment of textiles, mirrors, windshields, wallpaper, etc. with low-molecular-weight organosilicon compounds renders their surfaces water-repellent. Adaptation of these resins to standard laminating practice in the preparation of glass-fabric panels was announced in 1946.

Unsaturated Polyester Resins

These thermosetting resins differ from the types previously described in the nature of the curing or hardening reaction. The four previous types have involved a condensation reaction in which water is split off during the process of forming the resin. The unsaturated polyesters undergo a true polymerization in which combination of the monomers occurs through unsatisfied carbon double bonds. No water or other chemicals are split off and hence these resins can be processed at low pressures. These low-pressure resins removed the size limitations that presses and steel molds previously placed upon molded plastics applications and made possible the economical production of small numbers of pieces. This type of plastic is also relatively new, dating from about 1942.

The polyesters are used with glass fibers and fabrics to make strong, tough reinforced plastics. The strength and rigidity vary with the type of reinforcing material. Polyester laminates are resistant to most chemicals and have good weathering qualities. Rapid advances were made in this field under the stimulus of wartime requirements for aircraft parts, radar domes, and other military items, which were readily fabricated using these resins with various reinforcing materials, particularly glass fabric and mat. Plastics of this type are used in lightweight truck bodies, prefabricated housing panels, ducts for air-conditioning systems, luggage, and other products of large bulk, such as boats.

Epoxy Resins

These resins are based on reactions of ethylene oxide or its homologs or derivatives with any compound having groups containing active hydrogen, such as amino, carboxyl, and hydroxyl groups. The manufacture of epoxy thermosetting resins in the United States dates from about 1949; production capacity is now estimated to be about 45 million pounds.

Epoxy resins have an excellent balance of outstanding properties which accounts for their wide use in reinforced plastics, surface coatings, and adhesives. Organic bases, acid anhydrides, and compounds containing active hydrogen cure these epoxy resins without the formation of volatile by-products and with very little shrinkage. Therefore, these resins yield adhesives that

bond well to metals, glass, and ceramics. Another important factor in their use as adhesives, laminating resins, and potting compounds is that they can be cured at room temperature with only contact pressure. The cured resins have good mechanical and dielectric properties and are resistant to most common chemicals. Epoxy resins yield glass-fabric laminates that have wet flexural strengths of over 40,000 p.s.i. after 24 hr. at 500° F.

Isocyanate Resins

This new group of thermosetting resins is rapidly becoming of major commercial significance in the United States. The isocyanate compounds are commonly reacted with the hydroxyl groups of glycols or polyesters to produce polyurethanes. These can be made into fibers, molding compounds, foamed plastics, adhesives, elastomers, textile treatments, and coatings. Foamed products are the focal point of current developments in the utilization of the isocyanate plastics; they are being evaluated for many of the applications in which foam rubber is now used. Foamed-in-place isocyanate resins are available for insulating home freezers, water coolers, refrigerated trucks and cars, as a core material for radomes, and for buoyancy and flotation gear.

Thermoplastic Resins

There are six types of thermoplastic resins that are of major significance in the plastics industry. These are: 1) vinyl; 2) styrene; 3) polyethylene; 4) fluoroethylenes; 5) acrylic; 6) and nylon. Two additional types of thermoplastic resins, alkyd and hydrocarbon resins, are used mainly in protective coatings; in 1955 approximately 585 million pounds of alkyd and rosin-modified resins and 260 million pounds of hydrocarbon resins (coumarone-indene, polyterpenes, and petroleum resins) were produced for this purpose.

Vinyl Resins

Vinyl derivatives are basically those produced by the substitution of one of the hydrogens in ethylene by some other group; thus, the vinyl radical is $H_2C = CH -$. The first resins of this type appeared on the market in 1928. Their versatility has resulted in a steady increase in production and has made them the leader in annual volume of production; 650 million pounds of vinyl resins were produced in 1955.

The most important resins of this class industrially are the polymers and copolymers of vinyl chloride. They are used in wire and cable coatings, film for packaging, drapes, rain-wear, upholstery, floor tiles, and flexible tube for garden hose and laboratory purposes. Pipe made of unplasticized (rigid) polyvinyl chloride is proving useful in chemical plants as a replacement for special alloys to carry both acids and alkalies and for underground installations. The rigid sheet material is also being fabricated into tanks, baths, vats, and other chemical engineering equipment. Leading end uses of vinyl chloride plastics in 1955 are shown in Table IV.

Three other major vinyl resins are polyvinyl acetate, polyvinyl butyral, and polyvinyl alcohol. Polyvinyl acetate is used primarily in adhesives and coatings for paper, textiles, and leather. Polyvinyl butyral is the plastic used as the interlayer material in laminated safety glass for aircraft and

automotive glazing. Polyvinyl alcohol is made into hose, gloves, and protective clothing resistant to oils, gasoline, and most organic solvents.

Styrene Resins

Polymerized styrene (polystyrene) is the oldest known synthetic resin, having been prepared in the laboratory in 1839. Its commercial production in this country began in 1930, but did not attain significant proportions until 1937. The estimated 1955 production is 535 million pounds.

The most significant properties of polystyrene are its low power factor and practically zero water absorption. These remarkable properties make styrene resins exceptionally well suited for radio-frequency insulation. The resin is resistant to moderately strong acids, alkalies, and alcohol, and is fairly heat-stable. Its low cost, side color range, and ease of injection molding have made polystyrene the leading thermoplastic molding material for use in a wide variety of applications, such as kitchen items, refrigerator food containers, battery cases, wall tile, and instrument panels.

A variety of styrene derivatives and copolymers were introduced on the American market during World War II. The primary objectives in the development of these materials have been to improve the heat resistance and impact strength of polystyrene. An outstanding material of this type is made by the copolymerization of a high proportion of styrene with butadiene, which is then blended with natural or synthetic rubber to provide high-impact-strength plastics. The applications of these tough, chemical-resistant styrene copolymers include textile spools, chemical buckets, photographic trays, piping, safety helmets, machine housings, shipping containers, refrigerator food containers, battery cases, wall tile, and the like. Leading end uses of styrene plastics in 1955 are shown in Table V.

Polyethylene

Commercial manufacture of this resin was initiated in this country in 1943 for its use as electrical insulation on wire and cable. It has reached third position in annual volume of production among thermoplastics and will probably be the leading commercial plastic by 1960. Estimated production in 1955 is 350 million pounds.

In addition to its excellent dielectric properties, it has unusually good resistance to water and penetration of moisture, gives a tough, flexible film without plasticizer, and has a low specific gravity, 0.93. Polyethylene is familiar in flexible ice cube trays, tumblers, carboys, squeeze bottles, pipe, tubing, meteorological balloons, and freezer bags. The latest development is a polyethylene of higher modulus and higher heat resistance (250° F. continuously), which promises to become an important material for the fabrication of ducting, hoods, tanks, hospital equipment, machine housings, and the like. It is said of this new material that it "represents as great a stride in extending the versatility of polyethylene as did the first high-strength steels in extending the adaptability of steel about 30 years ago" (3). Leading end uses of polyethylene in 1955 are shown in Table VI.

Fluoroethylene Resins

Polytetrafluoroethylene, marketed as "Teflon", was first made during the latter part of World War II for military applications. It is inert to all types

Table V - End Use Consumption of Styrene Plastics in 1955[a]

Use	Percentage
Molding materials	70
Refrigeration	15
Packaging	10
Wall tile	10
Housewares	8
Toys	7
Radios and television	5
Bathroom supplies	5
Miscellaneous	10
Protective coatings	16
Miscellaneous	14
Total	100

[a] Sources: U. S. Tariff Commission reports and Modern Plastics (January 1956 issue).

Table VI - End Use Consumption of Polyethylene in 1955[a]

Use	Percentage
Film and sheeting	37
Molding material	18
Exported material	11
Wire insulation	11
Pipe	8
Coatings	8
Containers	3
Miscellaneous	4
Total	100

[a] Sources: U. S. Tariff Commission reports and Modern Plastics (January 1956 issue).

of chemicals except molten alkali metals. It does not have a true melting point, but does undergo a solid-phase change at 620° F. with a corresponding sharp drop in strength. Because of its chemical inertness, heat resistance, and extraordinarily low coefficient of friction, it has found many uses in gaskets, packings for pumps, liners for heat-sealing machines, chemical-resistant coatings, and insulation in motors, generators, and transformers.

By replacing one fluorine atom with chlorine, chlorotrifluoroethylene polymer is obtained. This compound has good moldability which retaining general inertness and high and low temperature resistance. It is used for chemical-resistant films and coatings, parts for chemical equipment, and coaxial cable insulation.

Acrylic Resins

The best known member of this group is methyl methacrylate resin which is the glass-clear material used as transparent enclosures on airplanes since 1936 because of its light weight, ready formability, weathering resistance, and relative shatter resistance. Recent work at the National Bureau of Standards has shown that the impact resistance of this material can be remarkably increased by multi-axial stretching, thus making it suitable for use in place of the more costly and heavier laminated glazing in pressurized aircraft. The advantageous properties of this light-transmitting plastic have resulted in its use in skylights, window glazing in factories and public buildings, luminous ceilings, and store facades. Other applications of poly (methyl methacrylate) include lenses, edge-lighted signs, molded reflectors for indirect highway lighting, and automotive light covers.

Other types of acrylic resins are used in protective coatings, finishes for leather, impregnating agents for textiles, and adhesives.

Nylon Resins

The best known use of this polyamide is as a textile fiber, but it has also found many other applications since its introduction in 1938. Outstanding properties of nylon plastics are low density, slow burning rate, toughness, flexibility, abrasion resistance, and high tensile strength in oriented monofilaments. Nylon resins are used to make bearings, gaskets, slide fasteners, gears, valve seats, and wire coverings. As monofilament, they are used in brush bristles and fishing lines.

Cellulose Plastics

Several cellulose derivatives are used as bases for plastic compositions. These include cellulose nitrate, cellulose acetate, cellulose acetate-butyrate, cellulose propionate, ethylcellulose and methylcellulose. They are made by modifying cellulose (cotton or wood pulp) with chemicals. All the cellulose plastics are tough and thermoplastic and can be formed into nearly colorless, transparent products. They are available as sheets, rods and tubes of various colors and as molding compounds.

Cellulose-nitrate plastic (nitrocellulose), the oldest commercial plastic, was marketed in 1870. It consists of cellulose nitrate plasticized with camphor and other plasticizers. An undesirable property of this material is its low ignition temperature and high rate of burning. The ignition temperatures and rates of burning of other cellulose plastics are more nearly like

those of paper and cardboard. Except for methylcellulose, the cellulose plastics are insoluble in water although they do absorb appreciable amounts of moisture. Methylcellulose is soluble in water.

Cellulose plastics are used in the manufacture of containers, dial crystals, windows for transmission of ultraviolet light, wrapping materials, safety guards, tool handles, toothbrush handles, flashlight housings, drawing instruments, fountain pens, goggles, and household appliance parts. The base of safety photographic film is cellulose acetate. Ethylcellulose and cellulose nitrate are used extensively in the formulation of protective coatings. An important application of cellulose acetate butyrate is in piping for irrigation, water disposal, and gas service lines.

Applications

It is evident that a discussion of applications of plastics has even more ramifications than consideration of the materials themselves. There is scarcely an industry that has failed to utilize these modern materials for either improving performance or streamlining appearance. The steel rolling mills use plastic bearings because they outwear their metal counterparts on this grueling job; the packaging industry employs plastics very extensively because of their eye appeal.

The following is a partial list of manufacturing groups that are consumers of plastics:

Abrasives	Office supplies
Adhesives	Optical goods
Air conditioning	Paper specialties
Aircraft	Pipes
Automotive	Plumbing equipment
Cooking equipment	Plywood
Display fixtures	Premiums
Electrical	Printing
Floor coverings	Radio
Food products	Railroad
Furniture	Refrigerating equipment
Hardware	Scientific instruments
House furnishings	Sporting goods
Luggage	Television
Machinery	Textiles
Marine	Tools

The growth of some of these markets during the ten-year period 1946-1955 is indicated in Table VII.

A symposium in September 1954, sponsored by the American Chemical Society, focussed attention on applications of plastics in chemical plants and equipment. Lightness of weight coupled with resistance to corrosion and ease of fabrication into complex shapes were cited as properties of particular interest to the chemical engineer. Plastic components in use include bearings, gears, rollers, guides, rings, packings, gaskets, piping, valves, ducts, kettles, tanks, and tubes. Applications of such equipment items in heavy and fine chemical plants, petroleum production, food processing, and textile manufacturing were described.

Table VII - Ten-Year Growth of Market for Products Using Plastics in Large Quantities[a]

Products	1946	1951	1954	1955(Est.)
Air conditioners (retail sales)	30,000	237,500	1,230,000	1,300,000
Floor-type vacuum cleaners (retail sales)	2,289,000	2,729,100	2,650,000	3,200,000
Television sets (production)	7,000	5,384,800	7,346,000	8,800,000
Refrigerators (retail sales)	2,100,000	4,075,000	3,593,000	4,000,000
Radios:				
Home (retail sales)	All types of radios 14,031,000	6,751,400	3,140,000	3,100,000
Portable (retail sales)		1,332,990	1,525,000	2,000,000
Clock (retail sales)		777,155	1,750,000	2,100,000
Washington machines (retail sales)	2,124,000	3,384,700	3,610,400	4,000,000
Automobiles (factory sales)	2,148,699	5,336,935	5,352,353	7,800,000
Trucks and buses (factory sales)	940,851	1,428,328	843,450	1,100,000
Dwelling units (non-farm)	671,000	1,091,300	1,220,400	1,100,000

[a] Source: Modern Plastics (January 1956 issue).

The chairman (E. B. Cooper) of the above-mentioned ACS Symposium on Plastics as Materials of Construction (4) noted that "it is significant that plastics frequently make their entry into an operating plant through the maintenance section, whose task it is to keep the plant operating. When other materials of construction prove inadequate, maintenance engineers may turn, in desperation, to plastics to keep the wheels turning. Designing plastics into new plants, however, requires engineering data, ensured sources of supply, and specifications—all well established for metals, ceramics, and concrete, but in early stages of development for most plastics. Designers of chemical plants cannot wait for these to be completed. They must specify conventional materials, realizing that they may be doomed for early replacement while properly chosen plastics would be permanent and frequently lower in cost. These symposium papers emphasize the need for design data that can be used with assurance."

A conference of the Building Research Institute in October 1954 dealt with the physical and engineering properties of plastics pertinent to their use in building construction and summarized the specific uses of plastics in this field. The latter include glazing, interior illumination, thermal insulation, vapor seals, structural panels, piping, ducts, conduits, wall and floor coverings, and decorative elements. The applications of plastics in the building industry will be discussed in detail by Dr. Dietz in another paper of this symposium.

The characteristics of plastics most useful to the building industry were summarized (5) as follows:

"1. Adaptability to shop-made or factory-assembled units which can be assembled and installed quickly at the construction site.

2. Availability of durable, decorative plastic surfaces.

3. Design and construction possibilities of interesting light-transmitting uses for walls and roofs.

4. Availability of flexible and adaptable interior arrangements through designing with lightweight plastic panels.

5. Ability of plastics to resist corrosion, weathering and wear for selected uses in building.

6. The tremendous opportunity for improved and low-cost applications of plastics in the growing usage of mechanical and electrical equipment in buildings.

7. The combination of plastics with conventional materials of construction affords opportunities for a much wider range of desirable structural properties.

8. Safety and cost features in the proper use of plastics are of interest to the building trades for saving time, money and weight.

9. Opportunities for dynamic coloring through plastic materials are of great interest to designers."

The Building Research Institute conference pointed up the following major problems facing the plastics industry in extending the uses of plastics in building construction: 1) accumulation of performance data to meet considerations of various types of loading, temperature, durability, and design; 2) definition of requirements and preparation of standards and specifications for plastics as construction materials; and 3) the promulgation of codes, legislation, and regulations governing the use of plastics in building.

The uses of plastics in the transportation industry were reviewed by Fligor [6] at the Diamond Jubilee Meeting of the American Society of Mechanical Engineers in November 1955. The applications in the automobile have been both functional and decorative. The most spectacular was the use of glass-polyester reinforced plastic for the body of the Chevrolet Corvette. The same material is being used to fabricate truck bodies and trailer tanks. A reinforced plastic milk truck tank 27 feet long, 5 feet high, and 18 feet in circumference has been produced with a weight saving of 50% over its metal counterpart. The marine transportation industry has utilized reinforced plastics in the construction of small boats. One boat builder predicts that within 5 years fifty percent of all boats under 16 feet in length will be made of plastics. The airplane industry employs over 11,000 parts made of plastics in the construction of aircraft. An outstanding application of reinforced plastics in an aircraft structural part is the 17 foot tail section of the Neptune patrol bomber which has proven to be economical because of cheaper tooling costs and reduced production time.

Fligor also strikes a note of caution with respect to the dearth of design data and standards for plastics. He notes that the designer has had many problems in adapting plastics due to lack of data on engineering properties for the many different plastic types. He continues: "Much of the published data have limited use because the creep tendencies of plastics have not been considered. Creep is the tendency of a material to deform under continuous load. It depends on factors, such as, temperature, kind of plastic, duration of load, nature of exposure, relation of load to the ultimate strength, etc. Intelligent design can only come after there has been time to relate service experience of materials to their engineering properties. Certainly the transportation industry will find prediction of service life of plastics quite complex because of the vibration forces found in vehicles and also the wide range of climatic exposure and the sudden nature of climatic change in transportation applications."

Standards

What is the present state of the development of standards for plastics suitable for use in civil engineering? Actually, only a small beginning has been made. Various subcommittees have been organized by Committee D-20 on Plastics of the American Society for Testing Materials, and by The Society of the Plastics Industry, Inc., to work on specifications for products for use in this field. Examples are the polyvinyl chloride plastic pipes and reinforced plastics made with polyester resins and glass fibers and fabrics [7].

The SPI also has sponsored investigations at Battelle Memorial Institute and the National Sanitation Foundation (headquartered at the University of Michigan's School of Public Health) to develop information on the properties of plastic pipes suitable for plumbing purposes, and it has groups working on the preparation of standards for plastic lighting fixtures, and foamed plastics.

The Monsanto Chemical Company has sponsored research at the Massachusetts Institute of Technology's School of Architecture on the use of plastics in building construction.

The National Electrical Manufacturers Association [8] and ASTM Committee D-20 on Plastics [9] have published specifications for decorative laminated plastics which are extensively used for wall paneling, window trim,

and the like, in public buildings, hotels, and restaurants.

Commerical standards for polystyrene wall tile and for colors for use in kitchen and bathroom accessories have been prepared by industry committees sponsored by SPI and the Manufacturing Chemists' Association, Inc., and published by the U. S. Department of Commerce (10). The Asphalt Tile Institute has issued a specification for plastic floor tile (11). The British Standards Institution also has published a number of standards covering plastic products used by the building industry (12).

The Department of the Air Force and the Navy Bureau of Aeronautics have contributed to the development of standards of interest to the building industry by their early and effective work on the preparation of specifications for reinforced plastics and the raw materials entering into their manufacture. Their work on specifications and utilization of transparent plastics has, likewise, been a contributing factor to the promising development in the use of these materials in industrial glazing (13).

NBS Work on Standards for Plastics

The National Bureau of Standards, in addition to maintaining the basic standards applicable to all the physical sciences and striving constantly to improve the accuracy and precision of such measurements, also has, as one of its statutory functions, the job of providing assistance to industry in the formulation of standards for products used in commerce and by government.

The NBS has participated actively in the preparation of these standards by the development of methods for determining the properties of plastics, by the determination and publication of technical data relating to many types of plastics, and by the assignment of personnel to work on technical committees engaged in the drafting of standards for plastics.

These committees have included those of the American Society for Testing Materials, the American Standards Association, International Standardization Organization, and The Society of the Plastics Industry, Inc., as well as various government committees (14).

The NBS is now cooperating with the various industry and technical society committees working on standards for plastics for the building industry. The pioneering efforts of these groups should lead to faster and surer progress in the application of plastics in civil engineering.

Properties of Plastics

The properties of plastics of interest to civil engineers are discussed in another paper presented as part of this symposium (15). Other surveys of the properties of plastics have been published by Dietz (16) and by the ANC-17 Panel on Plastics for Aircraft (17). Tables of properties of plastics are published annually in the Modern Plastics Encyclopedia (18, 19).

Thermoplastic materials are usually molded or cast as a homogeneous mass or sheet, without reinforcing agent or filler. These materials range in tensile strength from approximately 2,000 to 10,000 p.s.i. The initial tensile modulus of elasticity varies from approximately 200,000 to 400,000 p.s.i., and the proportional limit and yield strength are relatively low percentages of the ultimate strength. The compressive strength generally is of the same order of magnitude as the tensile strength, while the shear strength is lower. Other mechanical properties such as bearing strength and impact strength

are correspondingly low. The molecules in some thermoplastic materials such as nylon and vinylidene chloride can be oriented by stretching; filaments made in this way can attain a tensile strength approaching 100,000 p.s.i.

Molded thermosetting resins are usually reinforced with asbestos, cotton linters, cotton flock, macerated cotton, mica, wood flour, glass fibers. The tensile strength ranges approximately from 3,000 to 10,000 p.s.i. and the tensile modulus of elasticity is approximately 1,000,000 p.s.i.; the compressive strength is usually higher than the tensile strength. A wide range of impact values is obtainable, a glass fiber reinforced material having an Izod impact strangth of 12 to 24 foot-pounds per inch of notch. Properties also vary with the particular resin, molding conditions, and thickness of the molded sections. The complexity of the part has an effect on the properties of the material. This effect results from flow variations of the resin and reinforcing material, from the pressure on the various parts, from the rate of cooling, and so forth. Data obtained on test specimens molded under ideal conditions will not generally be representative of the properties of the material taken from various sections of a part molded in production.

Thermosetting resins have better mechanical properties when the reinforcing material is in a continuous form, such as paper, fabrics, and glass mats and filaments. Materials made by bonding together several layers of resin-impregnated reinforcing agent are called laminates. Very high tensile strengths have been obtained in laminates reinforced with glass fibers and fabrics, of the order of 40,000 to 50,000 p.s.i. The moduli of elasticity in tension in isotropic constructions range from 2,000,000 to 3,000,000 p.s.i.

Creep can be defined as the deformation of a material with time under the action of a constant stress. Thermoplastic resins are more susceptible to creep than are thermosetting resins. In the former materials, creep may occur at relatively low stresses, particularly at elevated temperatures; however, both types exhibit creep at room temperature. At tensile stresses up to 60 percent of the ultimate, most creep occurs during the first 100 hours, the total elongation for that time being less than 0.005 inch per inch for thermosetting laminates. The elongation per unit of time decreases and the curve then tends to level off. Creep can also occur under the action of flexural, compressive and shearing stressed combinations of stress.

If plastic materials are subjected to steady stresses approximating, but less than their static ultimate strength, they will fail after a relatively long period of time. Some comprehensive data on the long-time tensile strengths of various plastics have been published by Staff, Quackenbos, and Hill [20] and are presented in Tables VIII and IX.

The ANC-17 Bulletin [17] reports that creep parallel and perpendicular to warp of parallel or cross laminates at room temperature appears to be a negligible factor with glass-reinforced laminates. The creep in other directions may not be negligible. In addition, the creep may not be negligible for parallel laminates with unidirectional fabric except parallel to warp. In tensile and compressive tests of typical laminates, values of the ratio of initial to final deformation ranged from about 0.80 to 0.97, with most of the values being above 0.90. The lower values generally accompanied the longer loading periods (about 1,000 hours). Thus, with initial deformations on the order of 1 to 1-1/4 percent, the additional deformation due to creep is so small as to be negligible.

Table VIII.- Tensile Strengths and Moduli of Elasticity at 25° C. and 75° C. and Predicted Long-Time (5 year) Tensile Strengths of Phenol-Formaldehyde Materials[a] (Staff, Quackenbos, and Hill, reference 20)

Filler	25°C.			75°C.			Notch sensitivity[b]
	Tensile strength	Tensile modulus	5 year strength	Tensile strength	Tensile modulus	5-year strength	
	p.s.i.	10^6 p.s.i.	p.s.i.	p.s.i.	10^6 p.s.i.	p.s.i.	
Woodflour	7800	1.28	2000	8000	1.30	1200	2.0
Asbestos	4400	2.47	1500	4800	2.3	--	1.7
Chopped kraft paper	7700	1.42	2700	6500	1.39	1200	1.3
Macerated fabric	5600	1.49	2000	5600	1.45	1200	1.0
Cotton flock	7500	1.40	2700	6100	1.48	1200	1.4
Mica	5700	4.5	2000	6000	--	--	1.3

[a] The materials tested are commercial products. The type and content of resin varied. They were all virtually linear to failure in tension, and strain to failure can be found by dividing strength by modulus of elasticity.

[b] Notch sensitivity is the ratio of the breaking load in flexure for an unnotched bar 0.4 in. deep to the breaking load for a bar 0.5 in. deep containing the standard A.S.T.M. notch (0.1 in. deep, 0.01 in. radius) in the lower face.

Table IX.- Tensile Properties of Thermoplastics at 25° C. and Predicted Long-Time (1-Year) Tensile Strengths at 25° C. and 75° C.
(Staff, Quackenbos, and Hill, reference 20)

Material	25°C.				75°C.
	Tensile strength	Tensile modulus	Strain to failure	One-year strength	One-year strength
	p.s.i.	10^6 p.s.i.	%	p.s.i.	p.s.i.
Polystyrene	7500	0.48	2.1	800	250
Polymonochlorostyrene	9500	0.52	2.5	1200	400
Polydichlorostyrene	5900	0.60	1.0	1500	400
Styrene-acrylonitrile copolymer	9100	0.43	2.2	1500	560
Vinyl chloride-acetate copolymer	--	--	--	1600	--
Cellulose acetate[a]	6700	0.35	5.5	600	--
Cellulose acetate-butyrate[a]	3900	0.16	9.5	less than 600	--
Polymethyl-methacrylate[a]	10,100	0.43	4.2	2000	--

[a] Properties can vary with the mode of preparation. This is especially true with the cellulose plastics. "Medium" cellulose acetate and cellulose acetate butyrate were used here.

REFERENCES

1. Resources for Freedom, by the President's Materials Policy Commission, William S. Paley, Chairman. 5 vols. June 1952. Washington: Government Printing Office.
2. Past and Future Growth of the Chemical Industry, by R. H. Ewell, Chem. Eng. News 29, 5228 (Dec. 10, 1951).
3. The New Polyethylenes and Their Impact on Fabricated Parts, by J. A. Neumann and F. J. Bockhoff, Modern Plastics 32, 117 (Aug. 1955).
4. Plastics as Materials of Construction, Ind. Eng. Chem. 47, 1292-1367 (July 1955).
5. Summary of the Conference for the Plastics Industry, by R. K. Mueller. Chapter in Plastics in Building, p. 127. 1955. Washington: Building Research Institute, National Research Council.
6. Plastics in Transportation, by K. K. Fligor. Paper presented at meeting of A.S.M.E. Rubber and Plastics Division, Chicago, Ill., Nov. 17, 1955.
7. Annual Report of Committe D-20 of Plastics. Proceedings ASTM 54, preprint (1954). American Society for Testing Materials, 1916 Race St., Philadelphia 3, Pa.
8. Laminated Thermosetting Decorative Sheets. NEMA Publication No. LP2-1951 (June 1951). National Electrical Manufacturers Association, 155 East 44th Street, New York 17, N. Y.
9. Specifications for Laminated Thermosetting Decorative Sheets. ASTM D 1300-53T, 1954 Supplement to Book of ASTM Standards, Part 6. American Society for Testing Materials, 1916 Race St., Philadelphia 3, Pa.
10. Commercial Standards:
    CS147-47. Colors for Molded Urea Plastics.
    CS156-49. Colors for Polystyrene Plastics.
    SC168-50. Polystyrene Plastic Wall Tiles and Adhesives for Their Applications.
    CS181-52. Water-Resistant Organic Adhesives for Installation of Clay Tile.
    CS197-54. Dimensions and Tolerances for Flexible Standard Wall Polyethylene Pipe.
    The above are available from the Superintendent of Documents, United States Government Printing Office, Washington 25, D. C.
11. Specifications for Vinyl Plastic Asbestos Floor Tile. (August 16, 1951). Asphalt Tile Institute, 101 Park Ave., New York 17, N. Y.
12. British Standards:
    B.S. 1323: 1946. Synthetic-Resin Bonded-Paper Sheet (Thermosetting) for Use in the Building Industry.
    B.S. 1972: 1953. Polythene Tube for Cold Water Services.
    B.S. 1973: 1953. Polythene Tube for General Purposes Including Chemical and Food Industry Uses.
    The above are available from the British Standards Institution, 2 Park St., London, W 1, England.

13. Military Specifications:
MIL-R-7575. Resin, Low Pressure Laminating.
MIL-P-8013. Plastic Materials, Glass Fabric Base, Low Pressure Laminated, Aircraft Structural.
MIL-C-8073. Core Material, Plastic Honeycomb, Laminated Glass Fabric Base, for Aircraft Structural Applications.
MIL-P-9041. Plastic Material, Molded Sandwich Construction, Glass Fabric Base, Honeycomb Core, for Aircraft Structural Applications.
MIL-F-9084. Fabrics, Woven Glass, for Reinforced Plastic Laminates.
MIL-F-9118. Finish, Glass Fabric, for Reinforced Plastic Laminates.
MIL-P-5425A. Plastic; Acrylic Sheet, Heat Resistant.
MIL-P-6886. Plastic; Acrylic Sheet.
MIL-P-8184. Plastic; Acrylic Sheet, Modified.
The above are available from Commanding General, Wright Air Development Center, Wright Patterson Air Force Base, Ohio.

14. Federal Specifications:
L-S-137 (August 2, 1954). Screening, Plastic coated, Fibrous Glass, Insect.
L-S.139 (August 22, 1950) and Amendment-1 (August 29, 1952). Screening, Plastic, Insect.
L-S-751 (Interim, March 18, 1952). Tile; Floor, Vinyl Plastic.
The first two listed above are available from the Superintendent of Documents, United States Government Printing Office, Washington 25, D. C.; L-S-751 is available from General Services Adminstration, Federal Supply Service, Washington 25, D. C.

15. Plastics—Engineering Materials, by C. Howard Adams, ASCE Symposium on Plastics, Dallas Meeting, Feb. 17, 1956.

16. Physical and Engineering Properties of Plastics, by A. G. H. Dietz. Chapter in Plastics in Building, pages 11-20, published by Building Research Institute, Washington 25, D. C., (Apr. 1955).

17. ANC-17 Bulletin: Plastics for Aircraft. Part I. Reinforced Plastics (June 1955). Part II. Transparent Glazing Materials (in press). Published by Government Printing Office, Washington 25, D. C.

18. Plastics Properties Chart, insert in 1955 Modern Plastics Encyclopedia, published by the Plastics Catalogue Corp., New York 22, N. Y.

19. Laminates Chart, pages 866-869, 1955, Modern Plastics Encyclopedia.

20. Long-Time Tension and Creep Tests of Plastics, by C. E. Staff, H. M. Quackenbos, Jr., and J. M. Hill, Modern Plastics 27, 93 (Feb. 1950); Trans. A.S.M.E. 72, 697 (1950).

Journal of the
ENGINEERING MECHANICS DIVISION
Proceedings of the American Society of Civil Engineers

PLASTICS: ENGINEERING MATERIALS[a]

C. H. Adams[1]
(Proc. Paper 1072)

SYNOPSIS

Some read "Plastics—engineering materials" as a statement of fact, others as a question. It is the intent of this presentation to begin with the question and, by examining the characteristics of plastics from the civil engineer's viewpoint, to develop a case for saying that, in fact, plastics are engineering materials.

This paper is concerned solely with the engineering characteristics of plastics. It describes their reaction to the various environments of force, time, and temperature. The magnitude of the subject requires a generalized treatment such as would be used in a discussion of the engineering characteristics of metals. The subject will be developed through: 1) a series of comparisons of plastics with the older materials of civil engineering; 2) a series of comparisons of plastics by generic groups; 3) a selected set of quantitative comparisons of plastics with other materials; and finally, 4) an analysis of certain design considerations pertinent to plastics.

Civil Engineering

To establish the materials and by inference the properties of interest to the civil engineer, reference was made to the Encyclopedia Britannica. The Britannica, quoting from the Charter of the Institution of Civil Engineers (London) under date of 1828, has this to say about what civil engineering is, "(The) art of directing the great sources of power in nature for the use and convenience of man, as the means of production and traffic in states, both for external and internal trade as applied in the construction of roads, bridges, aqueducts, canals, river navigation, and docks for internal intercourse and exchange, and in the construction of ports, harbours, moles,

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breakwaters, and lighthouses and in the art of navigation by artificial power for the purposes of commerce and in the construction and adaptation of machinery and in the drainage of cities and towns." The civil engineer in the early nineteenth century was the only professionally recognized civilian engineer. He operated in many fields, some of which are today considered outside the province of the civil engineer. Today's civil engineer designs and builds heavy, durable structures. These include roads, bridges, dams, canals, and drainage systems for cities and towns. He is concerned then with strength, stiffness, and durability.

Comparison of Materials

The civil engineer in designing and building his roadway, bridge, or other structure uses many different types of materials. Four of these will be compared with plastics as a group of materials to bracket qualitatively the engineering characteristics of plastics. The data presented are on the essentially unoriented form of the material except for wood which is, of course, a directional product of nature.

The short time or so-called static fracture strengths of steel,[1] wood,[2] concrete,[1] and glass [3] are compared with those of plastics [4] in Fig. 1. Data are from tensile tests except those on concrete which are from compression tests. The strength range of plastics materials is broader by far than that of any one of the other materials shown except steel. Depending on the type of plastic composition and the reinforcement used, it is currently possible to cover over a two-hundred fold strength range. Most plastics are lower in strength than steel and the stronger woods. However, certain reinforced compositions approach steel in strength.

The elastic modulus of plastics (Fig. 2) is lower than that of steel and glass. They can be as flexible as low-modulus rubber or as stiff as concrete or wood. The plastics exhibiting the highest modulus of elasticity values, about 1/10th that of steel, are reinforced with glass fibers.

A comparison of the ductility of plastic materials with that of glass,[3] concrete,[1] and wood[5] shows that even the most brittle plastics[4] are generally more ductile than these. Their elongation at break in a tensile test measures from less than 1 to greater than 800 percent.

As a class of materials, plastics[4] are lighter than most of the conventional materials[1]. Depending on physical form, type of plastic material, and the reinforcing system used, plastics can weigh as little as 2 lb./cu. ft. to greater than 130 lb./cu. ft. The base polymers in solid form cover a much narrower range, from a specific gravity of 0.9 (lighter than water) up to 1-1/2 times as heavy as water.

The useful temperature range of the majority of today's plastics materials[4] is below the wood char point of approximately 380° F. (Fig. 3). Many materials in the thermoplastic (heat softening) group even in non-load bearing uses must be designed to function at less than the boiling point of water. The thermosetting (heat hardening) plastics can be used under the right circumstances at temperatures above the boiling point of water and even up to the wood char point previously mentioned. One or two types of plastics can be used under low load at 500° F.

Flammability is a property of significant importance in the engineering use of plastics. In general, plastics burn in a manner similar to wood. This

FIGURE 1

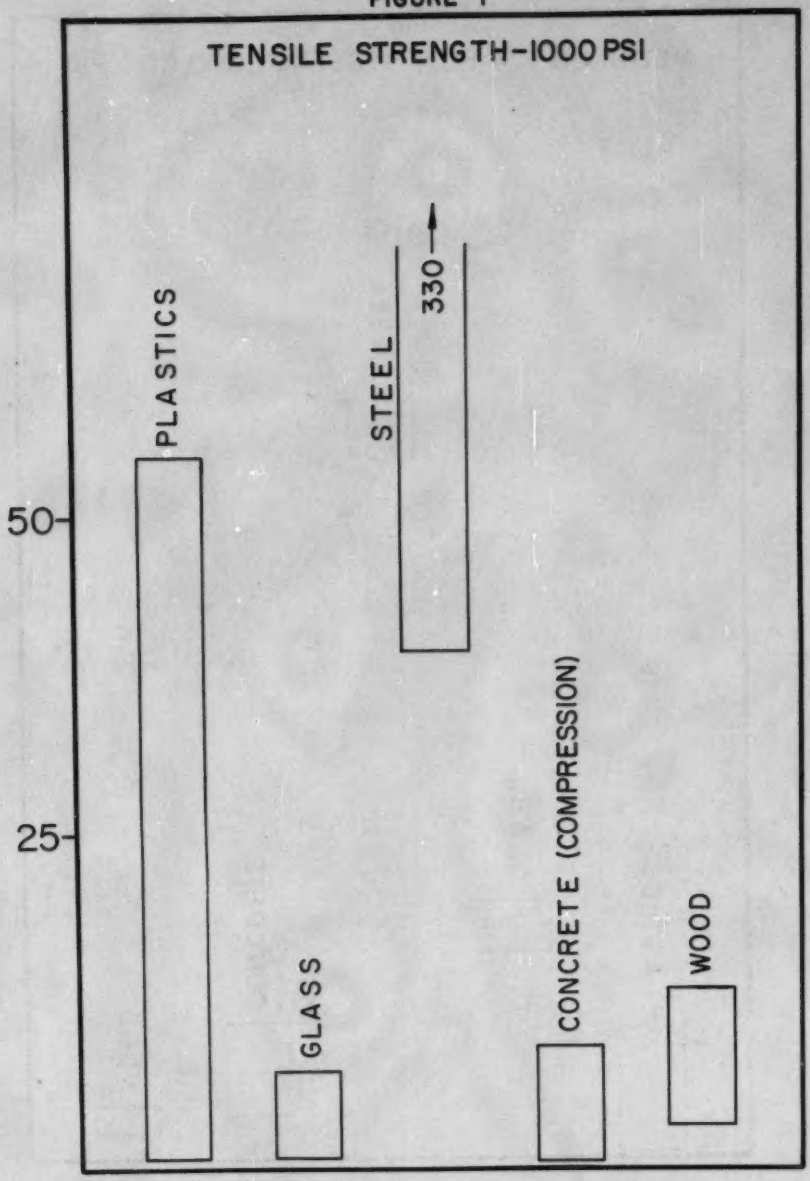

TENSILE STRENGTH-1000 PSI
50
25
PLASTICS
GLASS
STEEL
330
CONCRETE (COMPRESSION)
WOOD

FIGURE 2

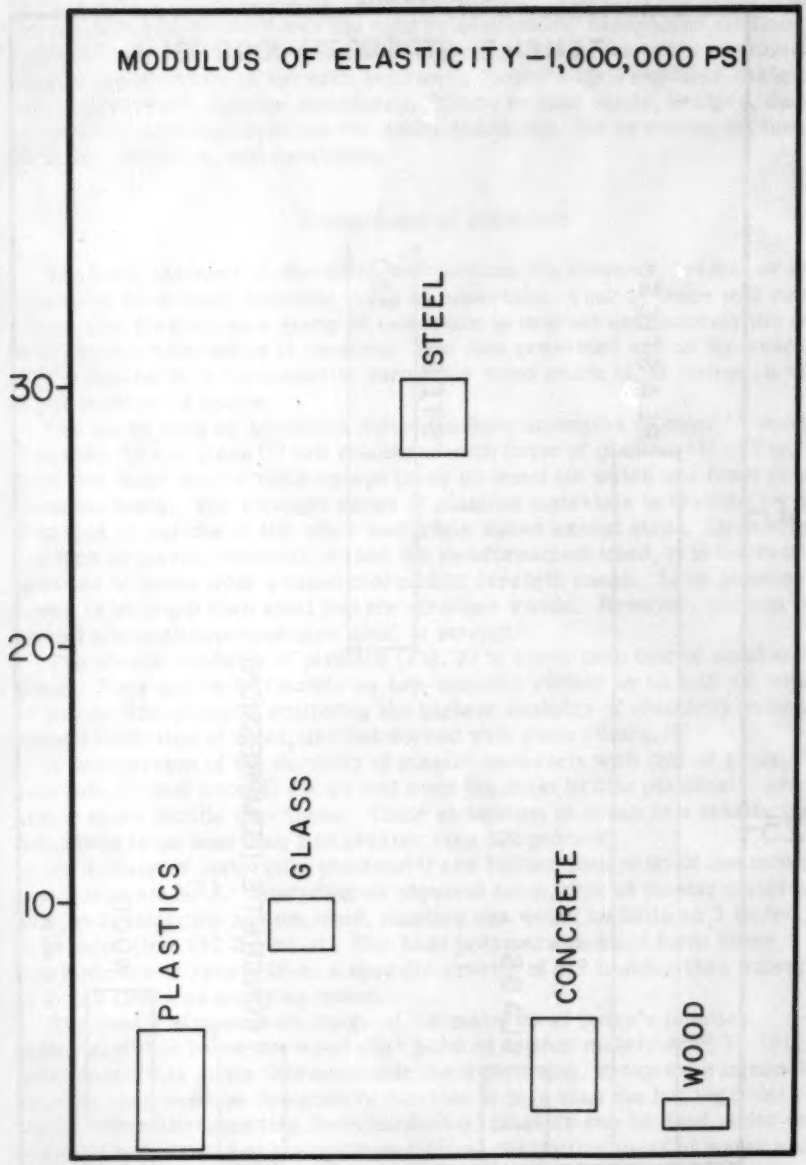
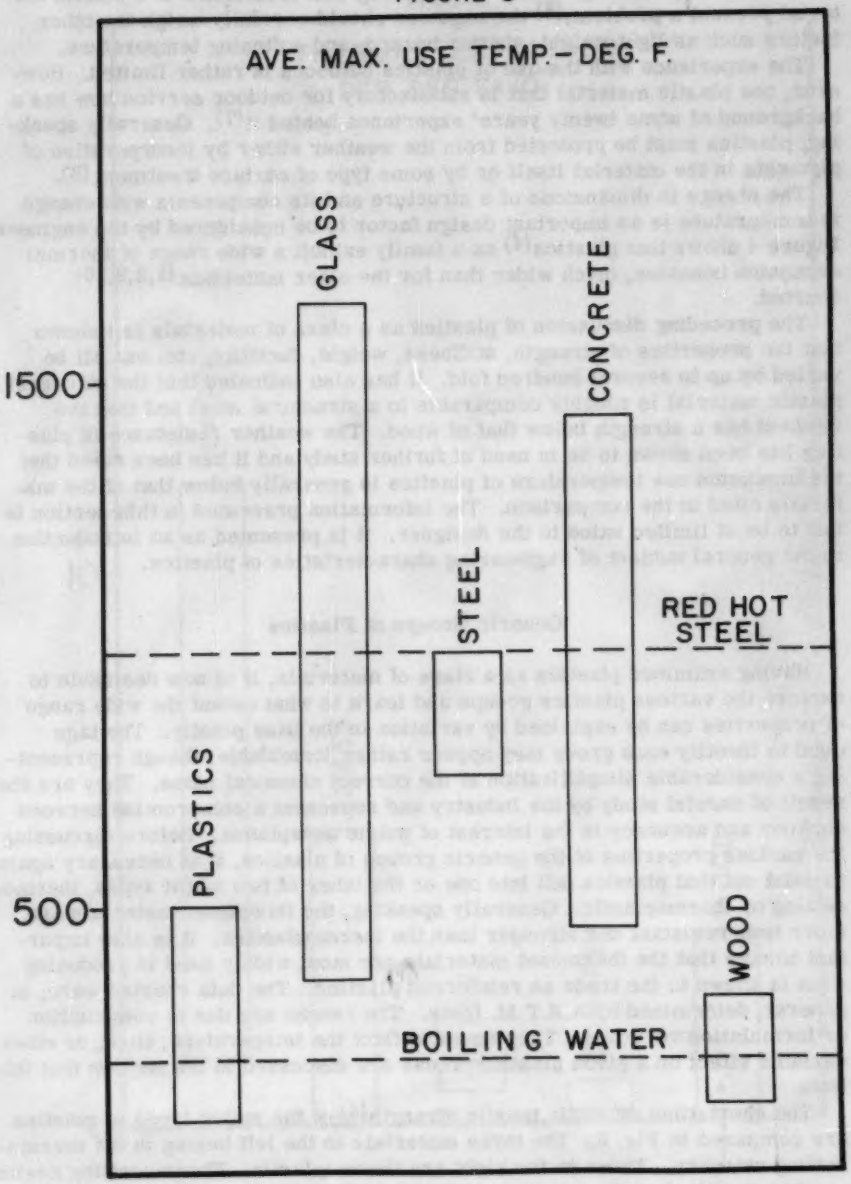

MODULUS OF ELASTICITY – 1,000,000 PSI
30
20
10
STEEL
GLASS
PLASTICS
CONCRETE
WOOD

FIGURE 3


AVE. MAX. USE TEMP.-DEG. F.
1500
500
GLASS
CONCRETE
STEEL
RED HOT STEEL
PLASTICS
WOOD
BOILING WATER

characteristic can be controlled to a substantial degree by composition modification. In considering whether the burning characteristics of a plastic material present a problem,[6] the engineer should carefully weigh the other factors such as light weight, shatter hazard, and softening temperature.

The experience with the use of plastics outdoors is rather limited. However, one plastic material that is satisfactory for outdoor service now has a background of some twenty years' experience behind it[7]. Generally speaking, plastics must be protected from the weather either by incorporation of pigments in the material itself or by some type of surface treatment [8].

The change in dimensions of a structure and its components with change in temperature is an important design factor to be considered by the engineer. Figure 4 shows that plastics[4] as a family exhibit a wide range of thermal expansion behavior, much wider than for the other materials [1,2,9,10] charted.

The preceding discussion of plastics as a class of materials has shown that the properties of strength, stiffness, weight, ductility, etc. can all be varied by up to several hundred fold. It has also indicated that the strongest plastic material is roughly comparable to a structural steel and that the weakest has a strength below that of wood. The weather resistance of plastics has been shown to be in need of further study and it has been noted that the maximum use temperature of plastics is generally below that of the materials cited in the comparison. The information presented in this section is felt to be of limited value to the designer. It is presented as an introduction to the general subject of engineering characteristics of plastics.

Generic Groups of Plastics

Having examined plastics as a class of materials, it is now desirable to explore the various plastics groups and learn to what extent the wide range of properties can be explained by variation in the base plastic. The tags used to identify each group may appear rather formidable, though representing a considerable simplification of the correct chemical name. They are the result of careful study by the industry and represent a compromise between euphony and accuracy in the interest of public acceptance. Before discussing the various properties of the generic groups of plastics, it is necessary again to point out that plastics fall into one or the other of two major types, thermosetting or thermoplastic. Generally speaking, the thermoset meterials are more heat resistant and stronger than the thermoplastics. It is also important to note that the thermoset materials are most widely used in producing what is known to the trade as reinforced plastics. The data charted were, in general, determined by A.S.T.M. tests. The ranges are due to composition or formulation variation. They do not reflect the temperature, time, or other variable effect on a given plastic. These are discussed in the section that follows.

The short-time or static tensile strength[4] of the major types of plastics are compared in Fig. 5. The three materials to the left belong in the thermosetting category. Those to the right are thermoplastic. Thermosetting resins combined with glass fibers give a high strength reinforced plastic breaking at a stress of 55,000 p.s.i. Thermoplastic resins such as vinyl are weakened by softening agents known as plasticizers.

FIGURE 4

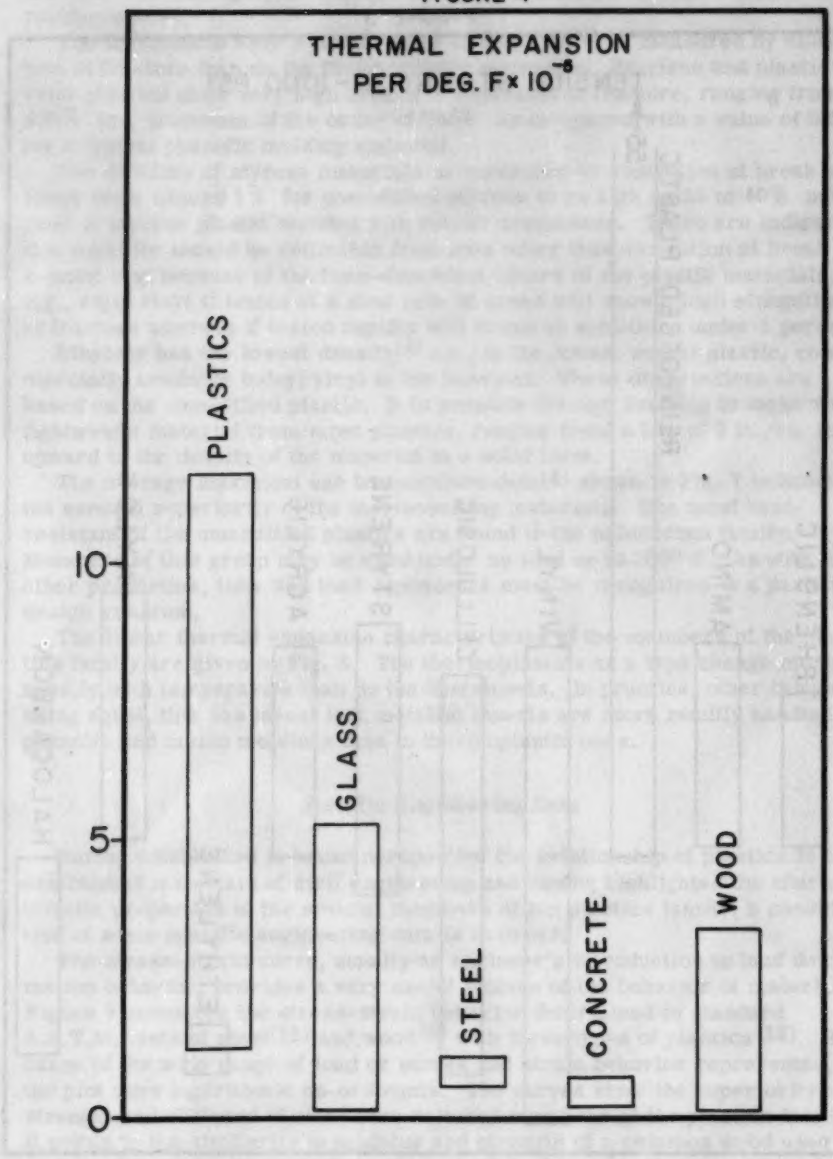
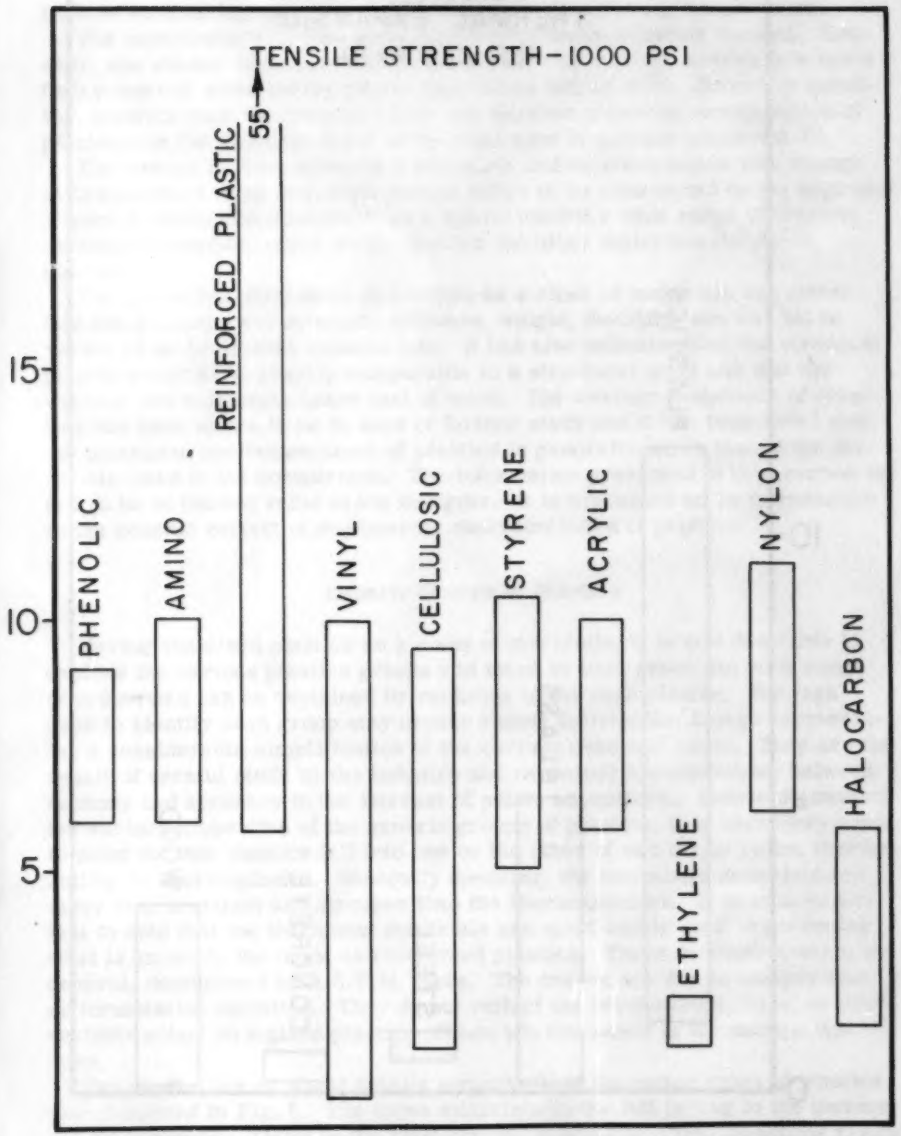

THERMAL EXPANSION
PER DEG. F x 10⁻⁶
PLASTICS
GLASS
STEEL
CONCRETE
WOOD
10
5
0

FIGURE 5


TENSILE STRENGTH-1000 PSI
55
15
10
5
PHENOLIC
AMINO
REINFORCED PLASTIC
VINYL
CELLULOSIC
STYRENE
ACRYLIC
ETHYLENE
NYLON
HALOCARBON

The stiffness characteristics[4] of plastics are indicated in Fig. 6. As was the case with strength, the highest stiffness is attained through the medium of reinforcement.

The thermosets show a lower order of ductility[4] as measured by elongation at fracture than do the thermoplastic materials. Ethylene and plasticized vinyl plastics show very high orders of extension at fracture, ranging from 300% to a maximum of the order of 700% as compared with a value of 0.5% for a typical phenolic molding material.

The ductility of styrene materials as measured by elongation at break will range from around 1% for unmodified styrene to as high as 35 to 40% in the case of styrene plastic blended with rubber compounds. There are indications that ductility should be estimated from data other than elongation at break in a static test because of the time-dependent nature of the plastic materials, e.g., rigid vinyl if tested at a slow rate of speed will show a high elongation at fracture whereas if tested rapidly will break at something under 1 percent.

Ethylene has the lowest density,[4] i.e., is the lowest weight plastic, commercially available today; vinyl is the heaviest. These observations are based on the unmodified plastic. It is possible through foaming to make very lightweight material from most plastics, ranging from a low of 2 lb./cu. ft. upward to the density of the material in a solid form.

The average maximum use temperature data[4] shown in Fig. 7 indicate the general superiority of the thermosetting materials. The most heat-resistant of the unmodified plastics are found in the halocarbon family. Members of this group may be used under no load up to 500° F. As with the other properties, time and load dependence must be recognized in a particular design situation.

The linear thermal expansion characteristics of the members of the plastics family are given in Fig. 8. The thermoplastics as a type change more readily with temperature than do the thermosets. In practice, other things being equal, this has meant that metallic inserts are more readily handled in phenolic and amino moldings than in thermoplastic ones.

Specific Engineering Data

Having established in broad perspective the relationship of plastics to the established materials of civil engineering and having highlighted the characteristic properties of the several members of the plastics family, a consideration of some specific engineering data is in order.

The stress-strain curve, usually an engineer's introduction to load deformation behavior, provides a very useful picture of the behavior of materials. Figure 9 compares the stress-strain behavior determined in standard A.S.T.M. tests of steel[11] and wood[5] with three types of plastics[12]. Because of the wide range of load or stress and strain behavior represented, the plot uses logarithmic co-ordinants. The curves show the superiority in strength and stiffness of steel over selected members of the plastics family. It points to the similarity in modulus and strength of a common wood used in construction and a laminated plastic material. Lower in the strength and stiffness scale are the vinyl plastic (rigid) and a low pressure ethylene plastic.

Creep data for certain plastics[13,14,15] and a typical mild steel[11] are compared in the semi-logarithmic plot of Fig. 10. The common stress,

FIGURE 6

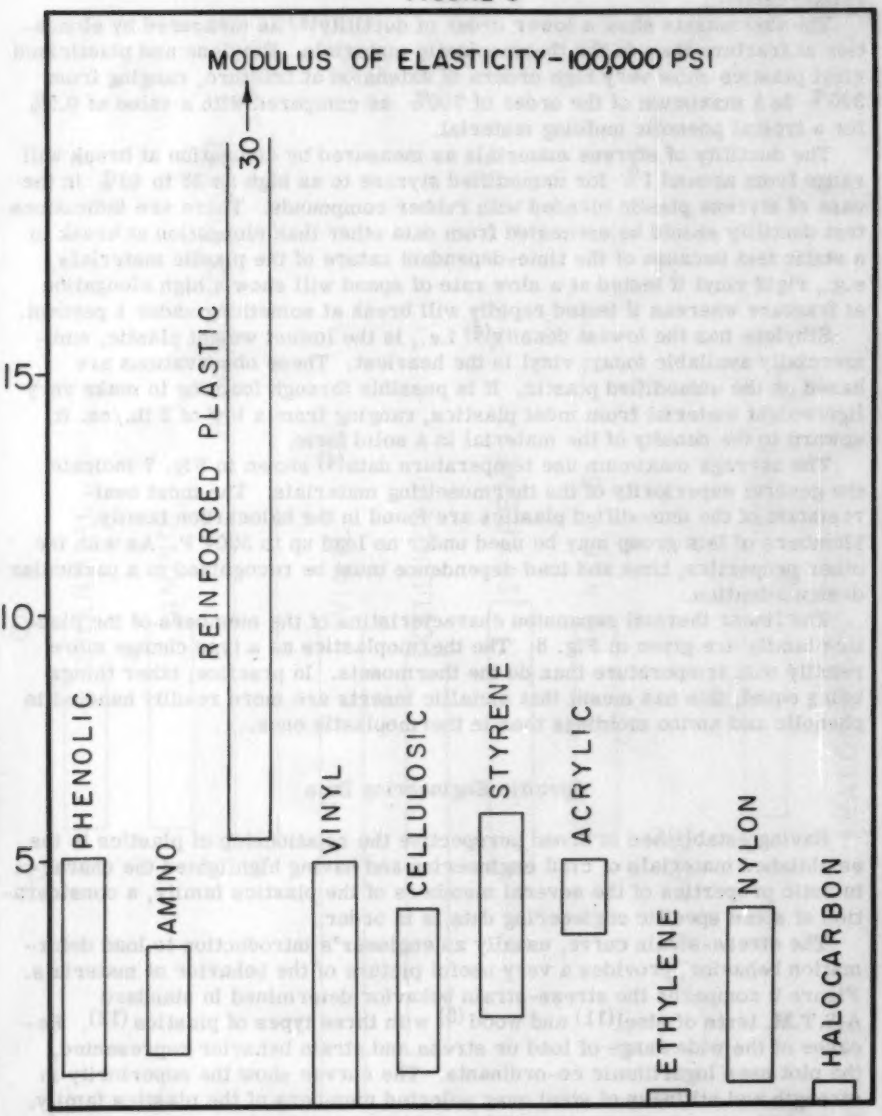

MODULUS OF ELASTICITY-100,000 PSI
30
15
10
5
PHENOLIC
AMINO
REINFORCED PLASTIC
VINYL
CELLULOSIC
STYRENE
ACRYLIC
ETHYLENE
NYLON
HALOCARBON

FIGURE 7

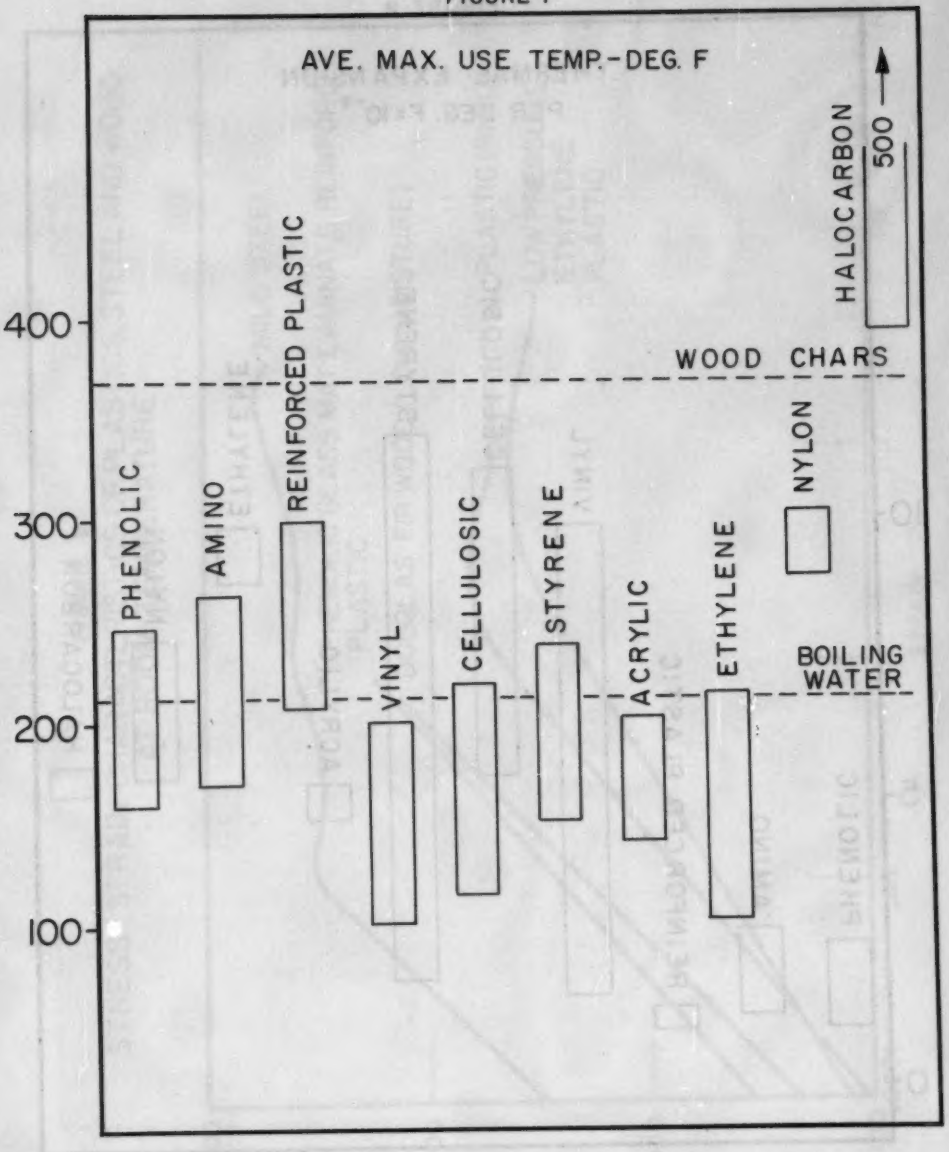

AVE. MAX. USE TEMP.-DEG. F
400
300
200
100
HALOCARBON
500
WOOD CHARS
REINFORCED PLASTIC
PHENOLIC
AMINO
VINYL
CELLULOSIC
STYRENE
ACRYLIC
ETHYLENE
NYLON
BOILING WATER

FIGURE 8

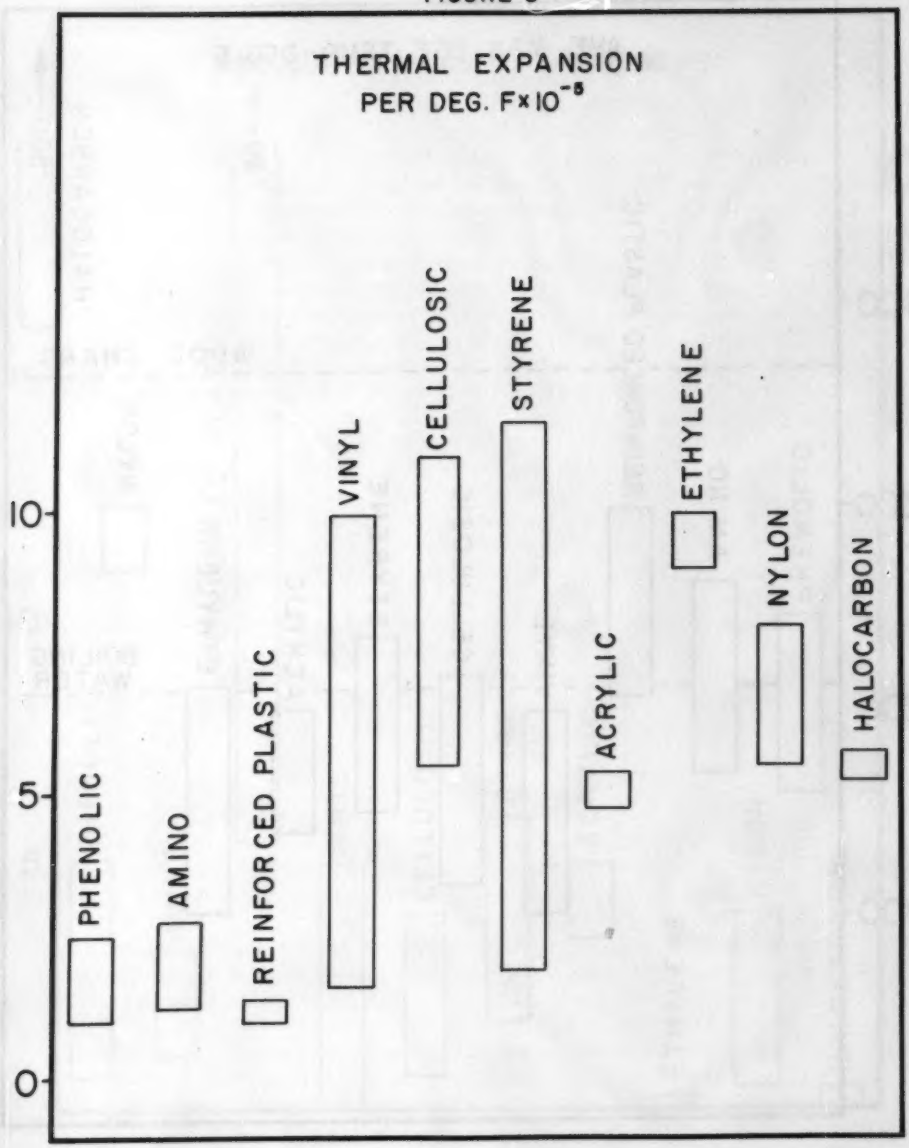

THERMAL EXPANSION
PER DEG. F x 10⁻⁵
0
5
10
PHENOLIC
AMINO
REINFORCED PLASTIC
VINYL
CELLULOSIC
STYRENE
ACRYLIC
ETHYLENE
NYLON
HALOCARBON

FIGURE 9

STRESS - STRAIN CHARACTERISTICS OF PLASTICS, STEEL AND WOOD AT ROOM TEMPERATURE

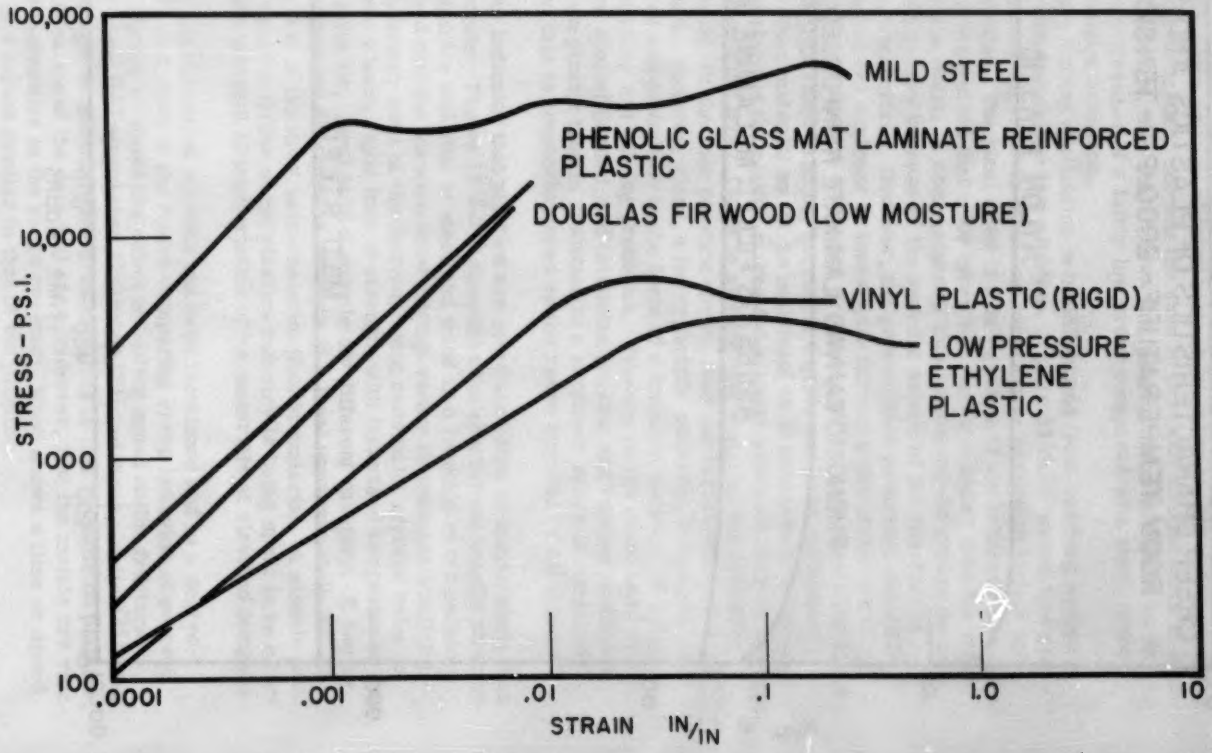
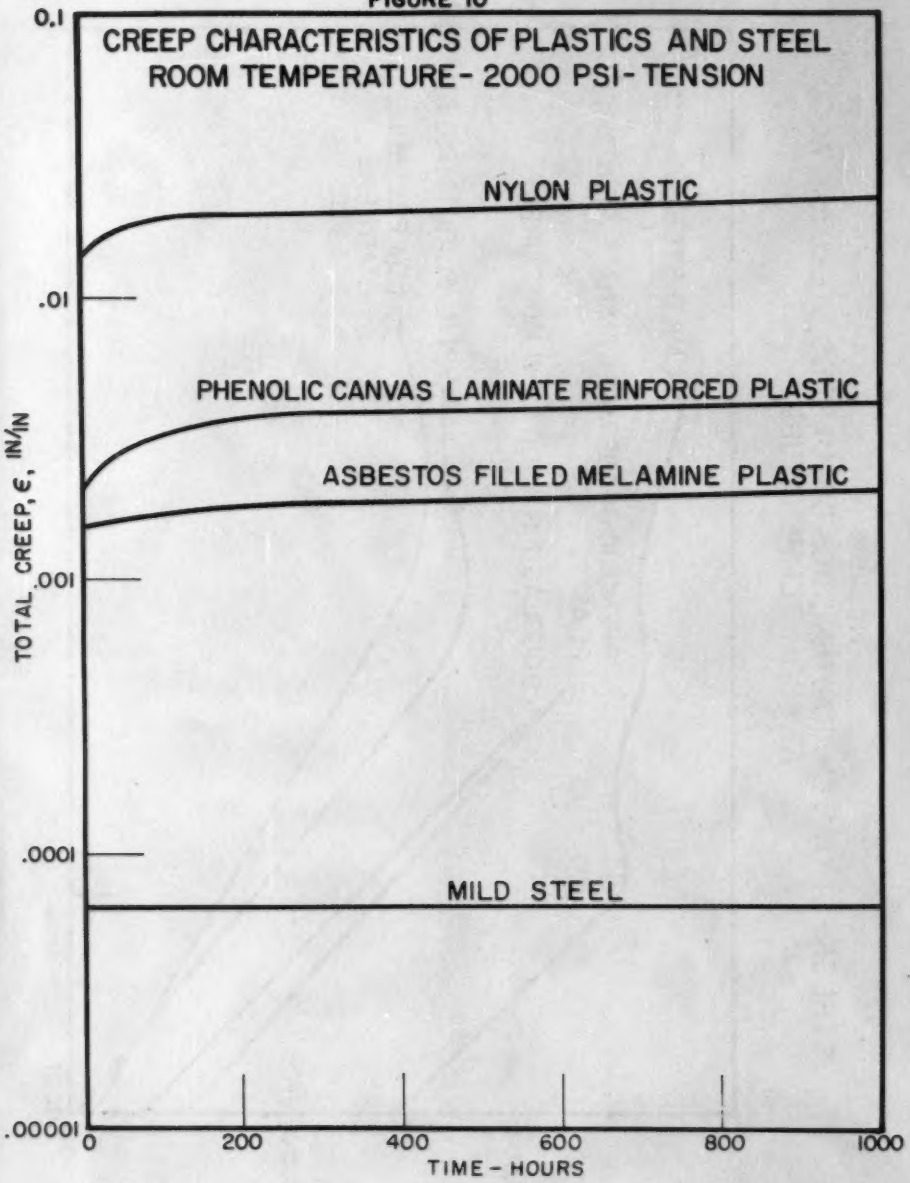


FIGURE 10


CREEP CHARACTERISTICS OF PLASTICS AND STEEL
ROOM TEMPERATURE - 2000 PSI - TENSION
NYLON PLASTIC
PHENOLIC CANVAS LAMINATE REINFORCED PLASTIC
ASBESTOS FILLED MELAMINE PLASTIC
MILD STEEL
TOTAL CREEP, ε, IN/IN
0.1
.01
.001
.0001
.00001
0
200
400
600
800
1000
TIME - HOURS

temperature, and loading conditions provide a basis for grading the load carrying ability of the materials shown. It is apparent that to design a part for a structure using any one of these materials, more specific data on creep behavior would be needed. As was the case in previous comparisons of load-deformation behavior, this chart shows that plastic materials must be designed to carry stresses of a fairly low order compared with steel, when creep is the design criterion.

Plastics are finding application in buildings and fluid-handling systems where corrosion resistance is a problem. Figure 11 (4,16) shows that plastics are generally superior to steel and wood where corrosive liquids and atmospheres exist. The steel cited in this figure is of the ordinary structural variety. It is recognized that alloy types will perform better than the material shown in the chart. A comparison of this nature should always be subject to critical review because of the specific nature of the reaction of chemicals with materials. However, the generalities presented should be helpful in leading the engineer to investigate further a particular plastic type when confronted with a corrosion problem. Certain plastics have found application in pipe lines handling corrosive liquids and have performed better than any other material as far as maintenance is concerned. In addition, the smooth surface of the plastic makes for a lower pressure drop in moving the liquid through the pipe. Glass is an excellent material for resisting chemical attack. Of the four plastics shown, only the halocarbon group is more resistant. However, glass, a heavy, brittle material, frequently requires support and protection in the form of a metallic backing. It tends to complicate jointing and valving problems. Plastics on the other hand are light, offer no shatter hazard, and minimize jointing and valving problems. Though a given plastic type is resistant to a narrower range of chemicals than is glass, this is counterbalanced by the range covered by all types of plastics.

It has been indicated that plastics are more sensitive to temperature than steel or concrete. Figure 12 (2,17) compares the short-time tensile strength of several plastics with that of steel and wood as a function of temperature. It will be noted that the plastics show a temperature dependence similar to that of wood, though most of the thermoplastic materials, styrene being an example, show a more rapid loss of strength with increased temperature than does the Douglas fir. Steel is in a completely different category. It begins to show a substantial decrease in strength at a level approximately 400 degrees above that of the most heat-resistant plastic material. A sizable portion of the research dollar of the plastics industry is being spent in an effort to upgrade the strength characteristics of its materials at elevated temperatures.

The time dependence of plastics has been mentioned and to a degree quantitatively tied down in the figure comparing creep behavior of several materials. Figure 13 shows the effect of testing speed on the fracture strength for steel,(18) magnesium,(19) wood,(20) and three plastics(21). The log-log plot tends to minimize the greater effect of this variable on plastics. Careful examination of the data will show, however, that the metals are relatively speed insensitive in the range covered. Wood shows a time or speed dependency of a nature similar to that of the plastics.

The data shown in the charts have come from many sources indicated in the references. Comprehensive source material on plastics is contained in

FIGURE 11

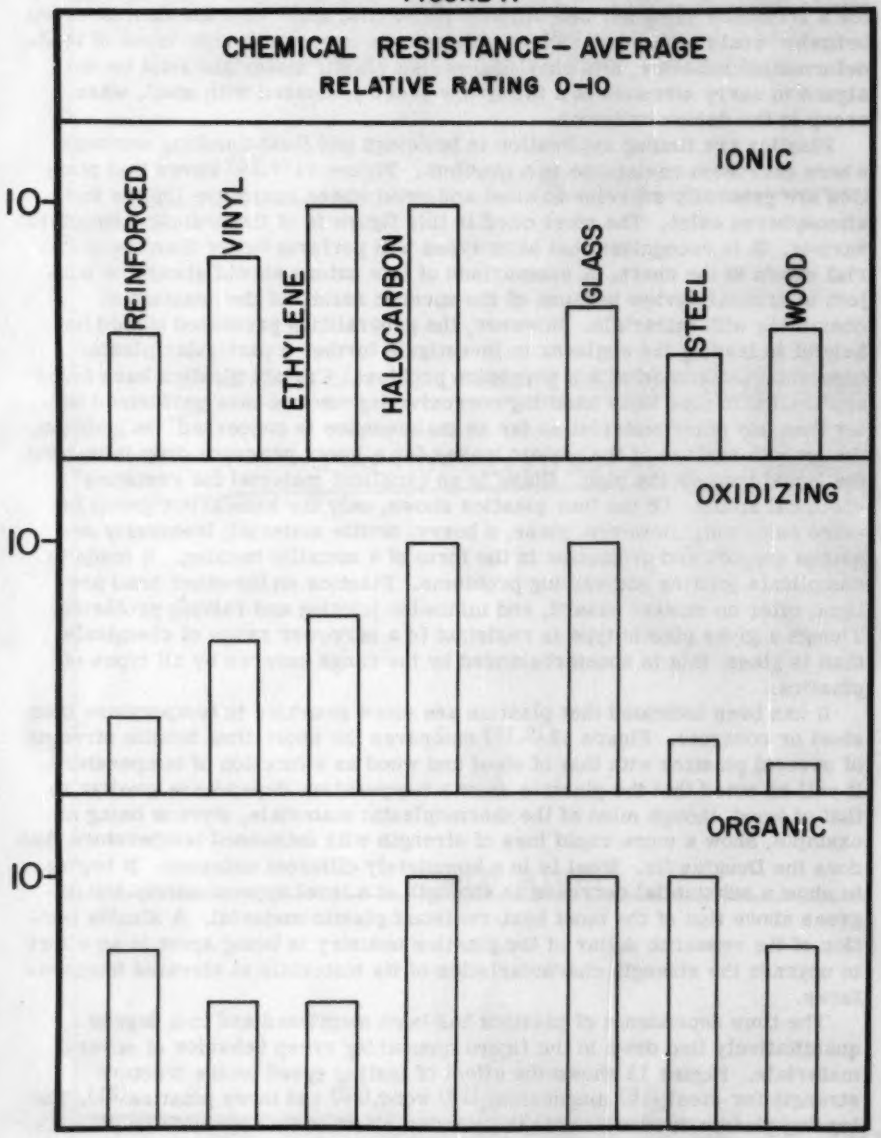

CHEMICAL RESISTANCE– AVERAGE
RELATIVE RATING 0–10
IONIC
10
REINFORCED
VINYL
ETHYLENE
HALOCARBON
GLASS
STEEL
WOOD
OXIDIZING
10
ORGANIC
10

FIGURE 12

TEMPERATURE DEPENDENCE, TENSILE PROPERTIES OF PLASTIC, STEEL & WOOD

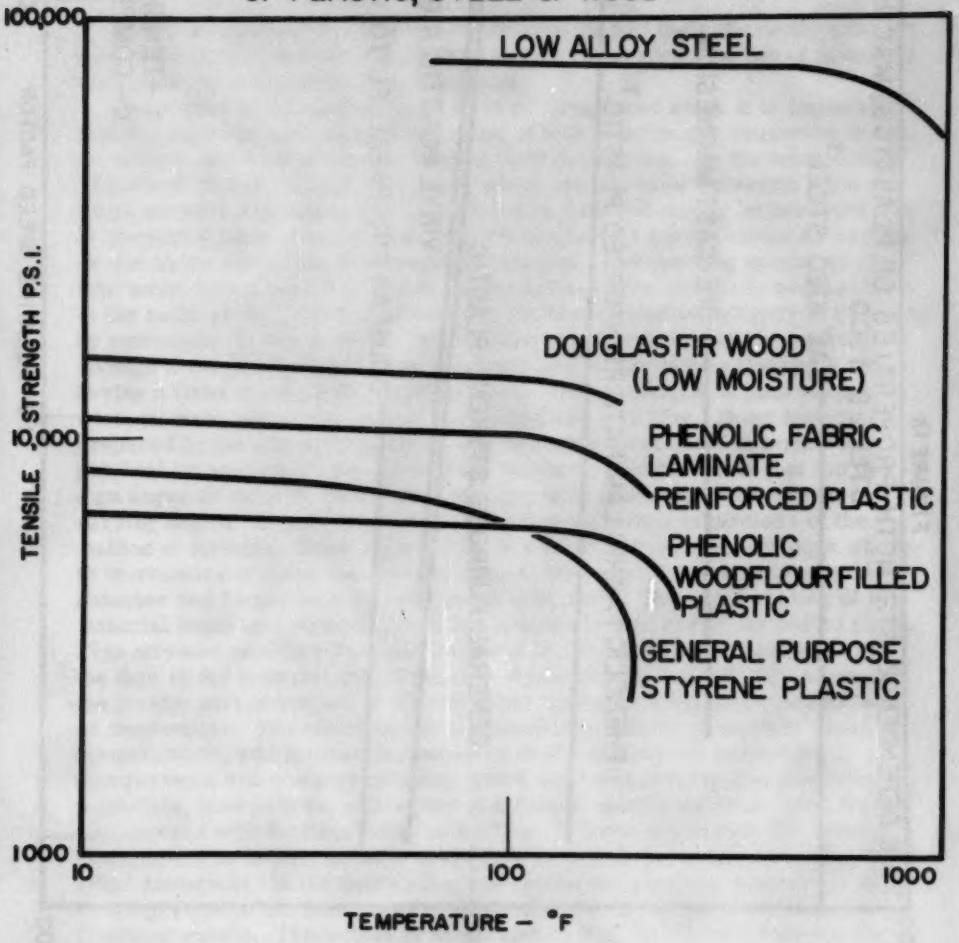
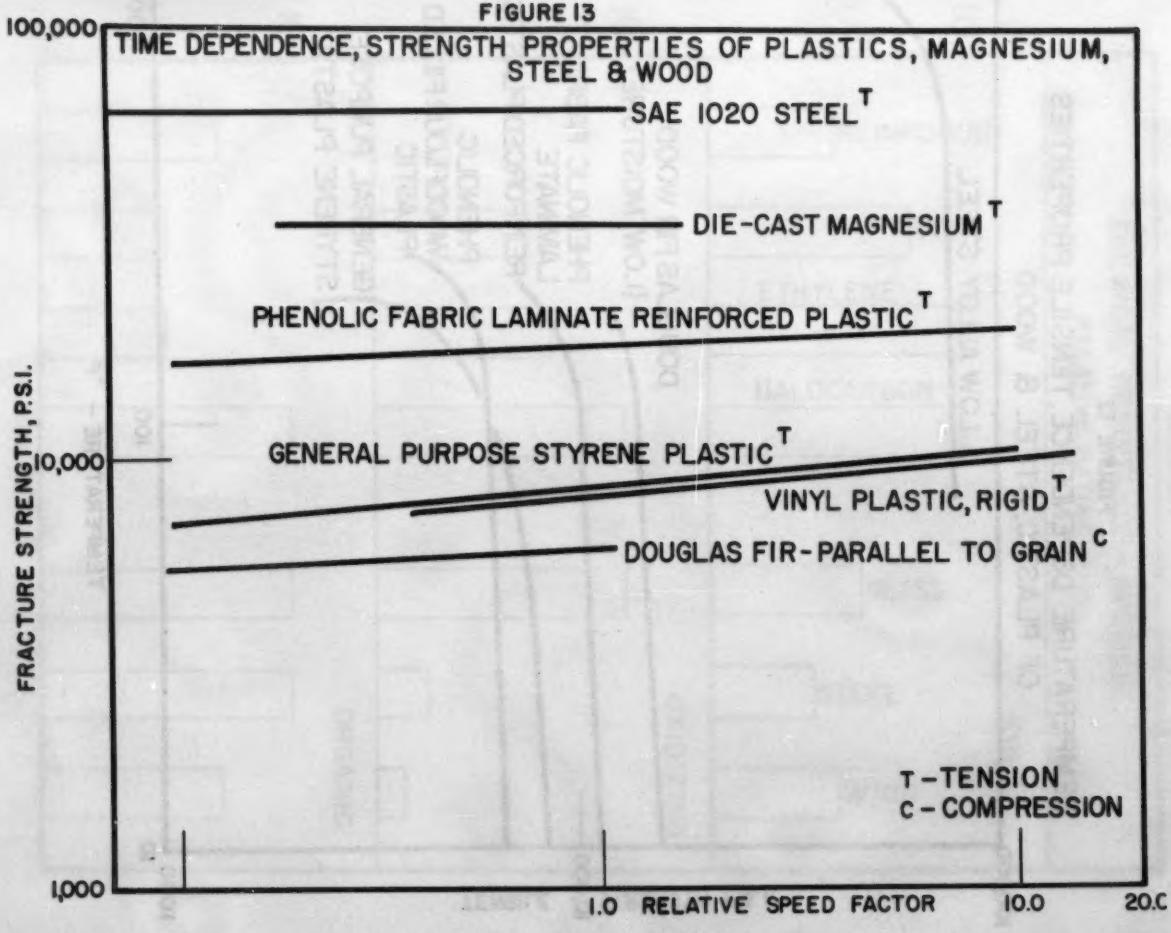


FIGURE 13


TIME DEPENDENCE, STRENGTH PROPERTIES OF PLASTICS, MAGNESIUM, STEEL & WOOD
SAE 1020 STEEL T
DIE-CAST MAGNESIUM T
PHENOLIC FABRIC LAMINATE REINFORCED PLASTIC T
GENERAL PURPOSE STYRENE PLASTIC T
VINYL PLASTIC, RIGID T
DOUGLAS FIR - PARALLEL TO GRAIN C
T - TENSION
C - COMPRESSION
FRACTURE STRENGTH, P.S.I.
100,000
10,000
1,000
1.0
RELATIVE SPEED FACTOR
10.0
20.0

the Manufacturing Chemists' Association book, "Technical Data on Plastics," the Society of the Plastics Industry's publication, "Plastics Engineering Handbook," the publications of the American Society for Testing Materials, and the various technical and trade journals devoted to materials and their behavior.

Design Considerations

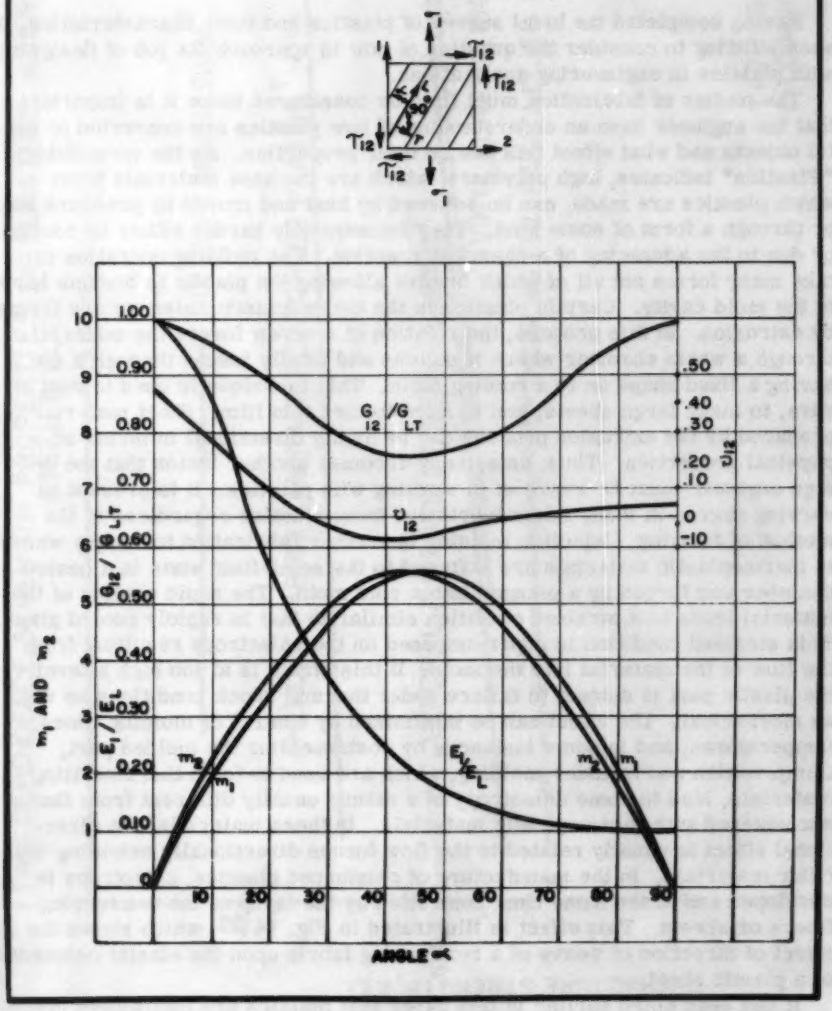
Having completed the brief survey of plastics and their characteristics, it seems fitting to consider the question of how to approach the job of designing with plastics in engineering applications.

The matter of fabrication must first be considered since it is important that the engineer have an understanding of how plastics are converted to useful objects and what effect this has on their properties. As the terminology "Plastics" indicates, high polymers, which are the base materials from which plastics are made, can be softened by heat and moved by pressure into or through a form of some kind. They subsequently harden either by cooling or due to the advancing of a chemical reaction. The molding operation can take many forms not all of which involve allowing the plastic to become hard in the mold cavity. Certain plastics in the thermoplastic category are formed by extrusion. In this process, the rotation of a screw forces the material through a warm chamber where it softens and finally issues through a die having a fixed shape on to a cooling table. This technique is used to coat wire, to make large sheets, and to manufacture thin film. Sheet material prepared by the extrusion process can be highly directional in terms of physical properties. Thus, anisotropy becomes another factor that the design engineer must be aware of in working with plastics. It is present to varying degree in many items fabricated from plastics regardless of the method of forming. Injection molding is another fabrication technique wherein thermoplastic materials are softened to the semi-fluid state in a heated chamber and forced by a plunger into a cold mold. The rapid cooling of the material leads to a strained condition similar to that in rapidly cooled glass. This strained condition is super-imposed on the anisotropy resulting from the flow of the material into the mold. If this strain is at too high a level, the plastic part is subject to failure under thermal shock conditions as well as mechanical. The effect can be minimized by control of molding times, temperatures, and in some instances by postannealing the molded part, Compression and transfer molding, which are used to form thermosetting materials, lead to some anisotropy of a nature usually different from that encountered with thermoplastic materials. In these materials, the directional effect is usually related to the flow forces directionally orienting the filler materials. In the manufacture of reinforced plastics, anisotropy is developed and at the same time controlled by the layup of the reinforcing fibers or sheets. This effect is illustrated in Fig. 14 [22] which shows the effect of direction of weave of a reinforcing fabric upon the elastic constants of a plastic sheet.

It has been noted earlier in this paper that plastics are lightweight materials. This has led some to the conclusion that plastics are the optimum materials for use in lightweight structures. This inference has been drawn in instances where the strength/weight ratio, i.e., the strength of a plastic divided by its weight, indicates that pound for pound the plastic material will

FIGURE 14

EFFECT OF DIRECTION OF WEAVE UPON ELASTIC CONSTANTS OF REINFORCED PLASTICS AND LAMINATES



give in some cases a stronger structure than steel. The fallacy of such a concept has been put in its proper perspective by several investigators. The finding of a group of these is summarized in Table I [23]. In this work, consideration was given to the problem of designing plastics structures for lightweight. It shows that the simple ratio of strength to weight is not satisfactory for predicting the minimum weight of a structure. It, in turn, emphasizes that structures perform many different functions. Some are designed primarily for strength; some, rigidity or stiffness; and some to damp vibrations. Depending on the particular design consideration superimposed on the main objective of a lightweight structure, a material heavier than a plastic may be used. Given the choices of type of loading and geometrical configuration of the structural member, the use of plastics in flexure as sandwich panels is favored. The sandwich panel design is based on the same principles of mechanics as the very familiar and widely used structural component, the "I" beam. In effect, the sandwich panel is an "I" beam with a very light web and high strength flanges. The web or core of a sandwich panel is made of a low density foamed plastic or from honeycombs of aluminum or plastic impregnated paper bonded with phenolic or other resinous adhesives. The flange or face is made of reinforced plastic, metal, or other material. This is bonded to the core with a plastic adhesive. This illustrates the important point that plastics in combination with other materials and in various forms give the engineer a new degree of design freedom.

The use of plastics as light transmitting materials in aircraft, automotive, and construction applications was pioneered by Henry Ford in his Model T. The reasons for using plastic in the T's side curtains were, no doubt, largely economic and its use was not subjected to critical engineering analysis. However, with the development of the acrylic aircraft canopy prior to and during World War II, plastics entered the light transmitting field for engineering reasons: e.g., the plastic was lighter than materials conventionally used, readily formed into complex shapes, and free of the shatter hazard of conventional glazing materials. The plastic allowed the aircraft designers considerable freedom and made possible a clean aerodynamic shape. Safety glass is made safe by virtue of a plastic interlayer between the glass surfaces. This contribution of plastics to safety began in the late twenties with the use of a cellulosic sheet which yellowed quickly and did not bond the glass faces too well. Continued research has led to improved vinyl type plastic interlayers. Panels of safety glass made with this type of plastic, exposed outdoors for periods in excess of ten years, show almost no color change and continued excellent adhesion to the glass faces. The use of plastic materials per se as automotive glazing materials is limited to the rear lights in convertibles because of the poor to far scratch resistance of most plastics. The convertible rear light has given the automotive designer more freedom than he had with glass sewed into the canvas back and at the same time improved the safety factor of convertibles by introducing a lightweight flexible material completely free of the shatter hazard.

Other areas with which the engineer must deal are fluid flow machinery, building components, machine components, electrical power transmission equipment, etc. It is hoped that the properties data and engineering analysis of the behavior of plastics presented herein will prove helpful to the civil engineer when faced with a material selection problem in one or more of his many fields of endeavor.

Table I

Comparison of Weights of Structures Made of Different Materials[a]

Functional specification	Criterion for minimum weight	Order of increasing weight of structure						
		SAE X4130 steel	Titanium alloy	75S-T aluminum alloy	Paper laminate, phenolic	Cast phenolic	Tough styrene	Crystal styrene
A. Strength								
1. Axial load	γ/σ	4 (2.83)	1 (1)	2 (1.12)	3 (1.81)	6 (5.12)	7 (9.18)	5 (4.68)
2. Bending of simple rectangular beam a. Width specified	$\gamma/\sigma^{1/2}$	7 (2.94)	3 (1.33)	2 (1.11)	1 (1)	5 (1.63)	6 (1.97)	4 (1.39)
b. Depth specified	γ/σ	4 (2.83)	1 (1)	2 (1.12)	3 (1.81)	6 (5.12)	7 (9.18)	5 (4.68)
3. Torsion of circular shaft	$\gamma/\tau^{2/3}$	5 (2.62)	—	1 (1)	4 (2.04)	3 (1.39)	—	2 (1.20)
B. Stiffness								
1. Axial load	γ/E	1 (1)	3 (1.08)	2 (1.04)	4 (2.25)	5 (7.17)	7 (10.6)	6 (10.1)
2. Bending of simple rectangular beam a. Width specified	$\gamma/E^{1/3}$	7 (2.40)	6 (1.70)	2 (1.22)	1 (1)	5 (1.41)	4 (1.40)	3 (1.35)
b. Depth specified	γ/E	1 (1)	3 (1.08)	2 (1.04)	4 (2.25)	5 (7.17)	7 (10.6)	6 (10.1)
3. Torsion of circular shaft	$\gamma/G^{1/2}$	5 (1.65)	— (1.30)	1 (1)	5 (1.66)	6 (1.77)	7 (2.24)	3 (1.62)
C. Resistance to impact Bending of simple rectangular beam	$\gamma E/\sigma^2$	7 (8.60)	1 (1)	2 (1.30)	3 (1.57)	5 (3.93)	6 (8.53)	4 (2.33)
D. Resistance to resonant vibration Axial load	$\gamma/CE\sigma^{n-1}$	2 (22.0)	—	—	1 (1)	4 (75.0)	5 (120.9)	3 (73.2)

[a] Numbers in parenthesis give relative weight of structure for material concerned in terms of weight of lightest structure.

Plastics are Engineering Materials

In the introduction the question was posed, "Are plastics engineering materials?" The data presented indicate that plastics have many useful properties which warrant their consideration as engineering materials. Among their useful characteristics are ease of fabrication, light weight, adaptability to combination with other materials to achieve new structural effects, and the ability to vary many of their properties over a wide range to suit the needs of a specific application.

REFERENCES

1. Moore, H. F., and Moore, M. B., "Textbook of the Materials of Engineering" 8th edition, McGraw Hill, New York, 1953
2. Agriculture Handbook No. 72, "Wood Handbook," Forest Products Laboratory, Madison, Wisconsin, 1955.
3. "Encyclopedia of Chemical Technology" (Kirkland and Othmer, Editors) Interscience, New York, 1951.
4. "Technical Data on Plastics," Manufacturing Chemists Association, Washington, D. C., October, 1952.
5. Dietz, A. G. H., "Stress-Strain Relations in Timber Beams (Douglas Fir)," ASTM Bulletin No. 118, 19, (Oct. 1942).
6. Rarig, F. J., "Building Code Regulation of Plastic Building Materials" Plastics in Building, Building Research Institute National Academy of Sciences, National Research Council, Publication 337, page 34, Washington, D. C., April, 1955.
7. Pierson, O. L., "Glazing and Interior Illumination," Plastics in Building, Building Research Institute, National Academy of Sciences, National Research Council, Publication 337, page 34, Washington, D. C., April, 1955.
8. Taylor, J. R., Adams, C. H., "Weather Aging of Styrene and Phenolic Plastics," Mechanical Engineering 76, 803, (Oct. 1954).
9. Materials and Methods 37, No. 5, 130 (1953).
10. Reinhold Publishing Corporation, New York, New York, "Portfolio of Material Engineering File Facts," 3rd Edition, 1951.
11. Schmidt, A. X., Marlies, C. A., "Principles of High Polymer Theory and Practise" McGraw Hill, New York, 1948.
12. Unpublished data, Plastics Division, Monsanto Chemical Company, 1956.
13. Findley, W. N., Adams, C. H. and Worley, W. J., "The Effect of Temperature on the Creep of two Laminated Plastics as Interpreted by the Hyperbolic-Sine Law and Activation Energy Theory" Proc. A.S.T.M. 48, 1217 (1948).
14. Marin, J., Webber, A. C., and Weissman, G. F., "Creep-Time Relations for Nylon in Tension, Compression and Bending" Proc. A.S.T.M. 54, 1313 (1954).

15. Telfair, D., Adams, C. H., and Mohrman, H. W., "Creep, Long Time Tensile and Flexural Fatigue Properties of Melamine, Phenolic Plastics," Modern Plastics 24, 151 (May 1947).

16. Seymour, R. B., "Practical Indices for Chemical Resistant Plastics," SPE Journal 10, No. 1, 15 (1954).

17. Nason, H. K., Carswell, T. S., and Adams, C. H., "Low Temperature Behavior of Organic Plastics" A.S.T.M. Special Technical Publication No. 78 (1949).

18. Jones, P. G., Moore, H. F., "An Investigation of Rate of Strain on the Results of Tension Tests of Metals," Proc. A.S.T.M. 40, 610 (1940).

19. Moore, A. A., "The Effect of Speed of Testing on Magnesium Base Alloys" Proc. A.S.T.M. 48, 1133 (1948).

20. Markwardt, L. J., and Liska, J. A., "Speed of Testing of Wood: Factors in its Control and its Effect on Strength" Proc. A.S.T.M. 48, 1139 (1948).

21. Dietz, A. G. H., Gailus, W. J., and Yurenlea, S., "Effect of Speed of Test Upon Strength Properties of Plastics" Proc. A.S.T.M. 48, 1160 (1948).

22. Dietz, A. G. H., "Design Theory of Reinforced Plastics," pp. 183-4, chapter in "Fiberglas Reinforced Plastics," R. H. Sonneborn, New York: Reinhold Publishing Corp.

23. Adams, C. H., Findley, W. N., and Stockton, F. D., "Considerations in the Design of Plastic Structures for Light Weight" Modern Plastics 32, 139 (Aug. 1955).

Paper 1073 EM 4

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Proceedings of the American Society of Civil Engineers

MINIMUM-WEIGHT DESIGN OF A PORTAL FRAME[a]

William Prager,[1] M. ASCE
(Proc. Paper 1073)

ABSTRACT

Plastic design of structural frames for minimum weight has been the subject of several recent papers. In these, the weight per unit length of a structural member has been assumed to be proportional to its fully plastic moment, though the effect of deviations from this simple relation has occasionally been discussed in a qualitative manner. The present paper is concerned with the minimum-weight design of frames under the assumption that the unit weight of a structural member is proportional to the α-th power of its fully plastic moment, the positive exponent α being smaller than unity. For $\alpha = 2/3$, a chart is developed that gives, at a glance, the minimum weight design for various geometries and loading conditions of a portal frame.

INTRODUCTION

Using concepts of limit analysis (see, for instance,(1)(2)(3)), several authors have recently explored the design of structural frames for minimum weight. Heyman(4) showed that the method of inequalities, which Neal and Symonds(5) had developed for the determination of the load carrying capacity of a frame of given dimensions, could be easily adapted to the problem of designing a frame for minimum weight, if the weight per unit length of a structural member could be taken as proportional to the fully plastic bending moment of this member. Foulkes(6) pointed out that under this assumption the minimum-weight design of a structural frame constituted a problem in linear programming (see, for instance,(7)). In a later paper,(8) the same author established necessary and sufficient conditions for a given design to be a minimum-weight design, assuming again that the unit weight of a member was proportional to its fully plastic moment. Making the same assumption,

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Livesley[9]* indicated a different approach and studied the programming of the problem for an automatic computer.

For most structural notions, the unit weight is more nearly proportional to a power of the fully plastic moment, with a positive exponent α that is smaller than unity. For example, $\alpha = 2/3$ if all available sections are geometrically similar. While Heyman and Foulkes have taken $\alpha < 1$ in a few simple examples, their general results are based on the assumption that $\alpha = 1$. It therefore seems worth while to reconsider the problem under more general assumptions regarding the relation between unit weight and fully plastic moment. The general discussion in Section 2 below is based on the assumption that the unit weights of the available structural sections are proportional to a certain power of their fully plastic moments, the positive exponent being smaller than unity. For the example of Section 3 this exponent has been taken as 2/3.

Theory

The plastic design of beams and frames is based on an idealized relation between bending moment and curvature. According to this relation, the curvature is negligible at all cross sections where the absolute value of the bending moment remains below a critical value M_0, while a yield hinge may develop at any cross section where the bending moment reaches this critical absolute value. This means that a beam or frame will remain practically rigid until yield hinges have developed at a sufficient number of cross sections to transform the structure, or some part of it, into a mechanism. The appearance of such a system of yield hinges is therefore taken as an indication that the practical load carrying capacity of the frame has been reached.

A system of infinitesimal displacements made possible by the insertion of an adequate number of yield hinges into the otherwise rigid members of the structure specifies a flow mechanism. Two systems of infinitesimal displacements that are obtained from one another by multiplication with a constant factor, are regarded as determining the same flow mechanism.

Given loads are beyond the load carrying capacity of a beam or frame, if there exists a flow mechanism for which the work of these loads exceeds the energy dissipated in the plastic bending at the yield hinges. Conversely, the absence of such a flow mechanism indicates that the given loads are within the load carrying capacity of the structure (see[2]). A flow mechanism for which the energy dissipated in the yield hinges equals the work of the given loads will be called a failure mechanism for these loads.

The following discussion is restricted to frames that consist of straight prismatic members subjected to concentrated loads only.** In a frame of this kind, yield hinges can occur only at loaded sections or end sections of members, and the number of possible flow mechanisms remains finite.

In the design problem considered here, the geometrical arrangement of

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* The author is indebted to Dr. R. K. Livesley for a copy of the manuscript of this paper.

** This does not imply a serious loss of generality because, for practical purposes, any curved member of varying cross section may be approximated by a polygon of straight prismatic members, and any distributed load may be approximated by an equipollent group of concentrated loads.

the members is given, and their fully plastic moments are to be determined so that (1) the given loads just exhaust the load carrying capacity of the frame and (2) the total weight of the frame is as small as possible, the unit weight w_i of a generic member being $w_i = cM_i^{\alpha}$, where C has the same value for all members, M_i is the fully plastic moment of the considered member, and the exponent α has the same value for all members and satisfies $0 < \alpha < 1$. We shall assume that n fully plastic moments are at the choice of the designer, though the frame may have more than n members. For instance, considerations of structural symmetry may bring about a reduction in the number of independent fully plastic moments.

For the geometrical discussion of the design problem, take the fully plastic moments M_i (i=1,2,. . . , n) as rectangular Cartesian coordinates in an n-dimensional space ("design space"). Any point ("design point") in the positive orthant ($M_i \geq 0$, i=1,2, . . , n) of this space then represents a specific design though not necessarily one that satisfies the conditions stipulated above.

For any flow mechanism, the condition that the work of the given loads must not exceed the energy dissipated in the yield hinges is expressed by a linear inequality in the fully plastic moments. Interpreted geometrically, such an inequality specifies a half space. The given loads exceed the load carrying capacity of any design represented by a design point outside this half space. The positive orthant of design space and the half spaces corresponding to the various flow mechanisms have a convex region in common. Only design points on the boundary of this region are admissible under condition (1) above. Since the number of possible flow mechanisms is finite, this boundary is a polyhedron with a finite number of (n-1)-dimensional faces. A design will be called admissible if it is represented by a design point on this polyhedral surface.

If ℓ_i denotes the combined length of all members with the fully plastic moment M_i, the structural weight of the frame is given by

$$W = c(\ell_1 M_1^{\alpha} + \ell_2 M_2^{\alpha} + \ldots + \ell_n M_n^{\alpha}). \qquad (1)$$

For a fixed positive value of W, this equation represents a surface in design space, which is convex with respect to the origin because $0 < \alpha < 1$. From this surface that corresponding to any other value of W is obtained by a similarity transformation with the origin as the center.

Figure 1 shows a typical situation for n=2. The curves correspond to various values of the structural weight W, and any point on the polygon ABCDE indicates a design whose load carrying capacity is just exhausted by the given loads. Since the polygon and the curves are convex with respect to the origin 0, the design of minimum weight must be represented by a vertex, viz. the point C in Fig. 1. As several vertices of the polygon may fall on the same curve of constant weight, the problem of minimum weight design may have several solutions.

Each side of the polygon corresponds to a failure mechanism in the following sense: with the exception of the endpoints, all points of a side represent designs that develop the same failure mechanism under the given load. A vertex, however, represents a design that may fail in either of the mechanisms corresponding to the adjacent sides, or in any positive linear combination of these mechanisms. Since the minimum weight design is represented by a vertex, it admits such a family of failure mechanisms.

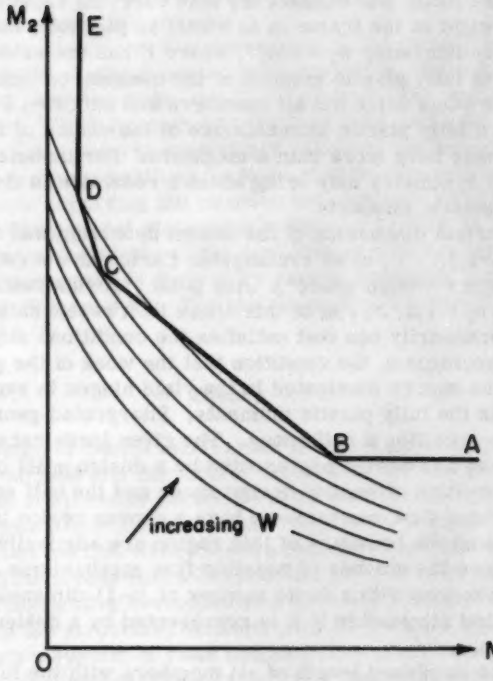

FIG. 1

In Fig. 1 any of these failure mechanisms is represented by a straight line through the considered vertex, which has only this point in common with the polygon. If the vertex represents a minimum weight design, these lines include the tangent to the curve of constant weight through the vertex. The equation of this tangent is of the form

$$M_1 \frac{\partial W}{\partial M_1} + M_2 \frac{\partial W}{\partial M_2} = \text{const.} \qquad (2)$$

where the derivatives must be evaluated at the given vertex.

For any failure mechanism, the balance between the work L of the given loads and the energy dissipated in the yield hinges is expressed by an equation of the form

$$M_1 \theta_1 + M_2 \theta_2 = L, \qquad (3)$$

where θ_i denotes the sum of the absolute values of the angle changes at all yield hinges with the fully plastic moment M_i. For the failure mechanism represented by the considered tangent the coefficients of the fully plastic moments in Eq. (3) must be proportional to the coefficients of these moments

in Eq. (2). On account of Eq. (1), this means that a minimum weight design admits a failure mechanism for which

$$\theta_1 : \theta_2 = \ell_1 M_1^{\alpha-1} : \ell_2 M_2^{\alpha-1}. \qquad (4)$$

While necessary, this condition is not sufficient for the considered admissible design to be a minimum weight design. For example, the design represented by the point B in Fig. 1 admits a failure mechanism satisfying (4), because the tangent of the curve of constant weight through B has only this point in common with the polygon. All the same, the structural weight of the design B exceeds that of the design C. The minimum weight design C is characterized by the following properties: (a) it admits a failure mechanism satisfying (4), and (b) there is no admissible design of lesser weight admitting such a failure mechanism.

The situation is very much the same for $n > 2$. Necessary and sufficient conditions for a design to be a minimum weight design are as follows:

a) the design must admit a failure mechanism satisfying

$$\theta_1 : \theta_2 : \ldots : \theta_n = \ell_1 M_1^{\alpha-1} : \ell_2 M_2^{\alpha-1} : \ldots : \ell_n M_n^{\alpha-1}; \qquad (5)$$

b) there must not be any admissible design of lesser weight admitting such a failure mechanism.

Example

Following Foulkes,[8] we consider as example the portal frame shown in Fig. 2 and propose to derive a design chart, which exhibits the dependence of the minimum weight design on the ratios h/ℓ and $(Qh)/(P\ell)$. The coefficient α in Eq. (1) will be taken as 2/3.

For positive values of the loads P and Q, only the flow mechanisms shown in Fig. 3 and their positive linear combinations need to be considered. To the mechanisms of Fig. 3 there correspond the following inequalities, which must

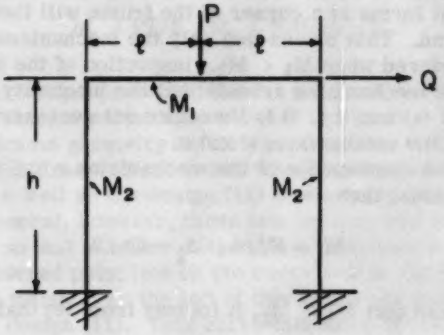

FIG. 2

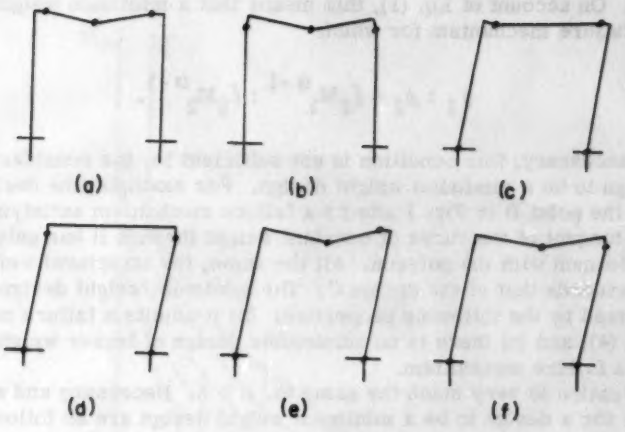

FIG. 3

be fulfilled if the given loads are not to exceed the load carrying capacity of the frame:

$$4M_1 \geq P\ell \quad , \tag{a}$$

$$2M_1 + 2M_2 \geq P\ell \quad , \tag{b}$$

$$2M_1 + 2M_2 \geq Qh \quad , \tag{c}$$

$$4M_2 \geq Qh \quad , \tag{d}$$

$$4M_1 + 2M_2 \geq P\ell + Qh \, , \tag{e}$$

$$2M_1 + 4M_2 \geq P\ell + Qh \, . \tag{f}$$

To discuss the import of these inequalities, assume first that $M_1 < M_2$. Any yield hinge that forms at a corner of the frame will then be in the beam rather than a column. This means that only the mechanisms a, c, and e (Fig. 3) need to be considered when $M_1 < M_2$. Inspection of the inequalities corresponding to these mechanisms reveals that the inequality (e) is more critical than the sum of (a) and (c). It is therefore not necessary to investigate the combination of the mechanisms a and c.

Consider next the combination of the mechanisms a and e. If both are to be failure mechanisms, then

$$M_1 = P\ell/4 \, , \; M_2 = Qh/2. \tag{6}$$

Since it was assumed that $M_1 < M_2$, it follows from (6) that

$$(Qh)/(P\ell) > 1/2. \tag{7}$$

With $\alpha = 2/3$, the condition (4) assumes the form

$$\frac{\theta_2}{\theta_1} = \frac{h}{\ell}\left(\frac{M_1}{M_2}\right)^{1/3} . \tag{8}$$

In the inequality (a) the ratio of the coefficients of M_2 and M_1, i.e. the ratio θ_2/θ_1, has the value 0; in (e) this ratio has the value 1/2. As the mechanisms (a) and (e) are combined with arbitrary positive coefficients, θ_2/θ_1 thus varies from 0 to 1/2. According to (8), we therefore have

$$0 \leq \frac{h}{\ell}\left(\frac{M_1}{M_2}\right)^{1/3} \leq 1/2. \tag{9}$$

Substitution of M_1 and M_2 from (6) into (9) furnishes

$$0 \leq \left(\frac{h}{\ell}\right)^3 \leq \frac{1}{4}\frac{Qh}{P\ell} . \tag{10}$$

In the design chart of Fig. 4, h/ℓ and $(Qh)/(P\ell)$ are used as rectangular coordinates. Since these are restricted to positive values, a specific geometry of the frame (h/ℓ) and type of loading (Q/P) are represented by a point in the first quadrant. The heavy lines divide this quadrant into regions labelled by the letters from A to G. For each of these regions, the minimum weight design is specified by the formula given for this region in the legend of Fig. 4.

Each of these regions corresponds to the combination of two of the flow mechanisms shown in Fig. 3. Region A, for instance, corresponds to the combination of the mechanisms a and e, which has been investigated above. According to the inequalities (7) and (10), region A cannot extend below the line $(Qh)/(P\ell) = 1/2$, to the left of the line $h/\ell = 0$ or to the right of the line $(H/\ell)^3 = (Qh)/(4P\ell)$. In Fig. 4 these are the lines 1-2, the vertical axis, and the curve 2-3.

Region B corresponds to the combination of the mechanisms c and e. If both are to be failure mechanisms, then

$$M_1 = P\ell/2 \ , \ M_2 = (QH-P\ell)/2. \tag{11}$$

A discussion similar to that presented for the region A shows that the region B cannot extend beyond the broken line 4-5-6-7 in Fig. 4.

Consider now a frame geometry and a type of loading that are represented by a point between the lines 4-5 and 3-10. The preceding discussion shows that the design (6) as well as the design (11) then admits a failure mechanism satisfying (4). In general, however, these two designs will have different structural weights, so that only one of them is a minimum weight design. Only when the considered point lies on the curve 8-9 do the two designs have the same structural weight. To the left of this curve the design (6) is lighter and to its right, the design (11). This curve therefore is the common boundary of the regions A and B. Its equation is found by setting the weight of the design (6),

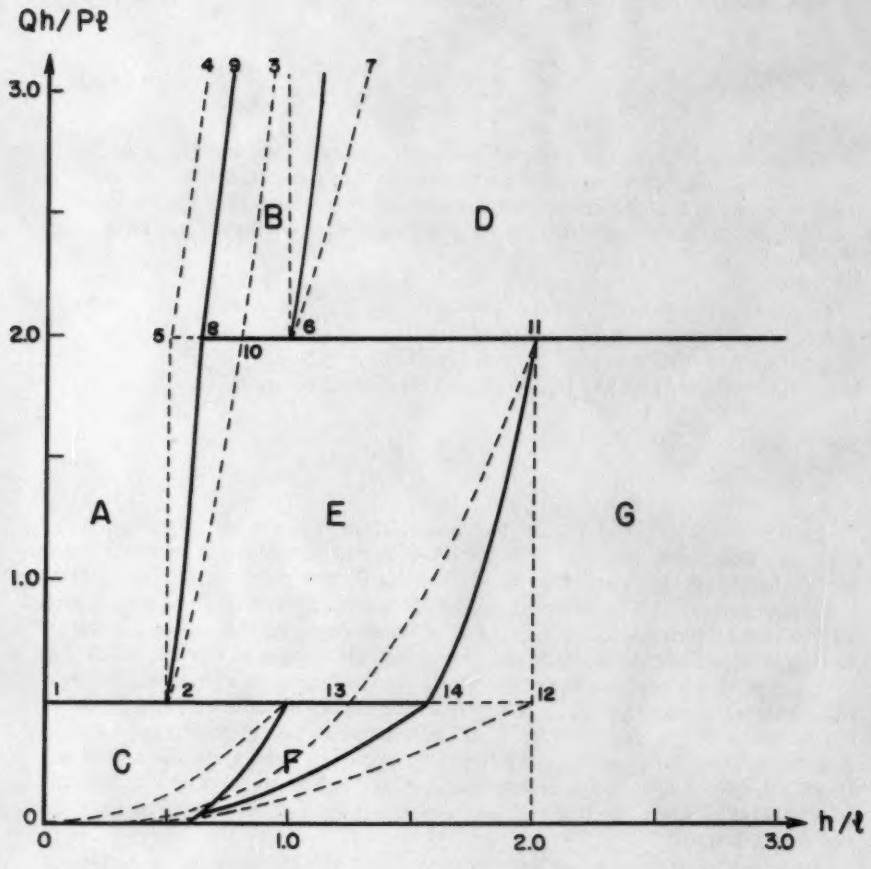

FIG. 4

Minimum weight design chart for portal frame

Region A: $M_1 = P\ell/4$, $M_2 = Qh/2$

Region B: $M_1 = P\ell/2$, $M_2 = (Qh - P\ell)/2$

Region C: $M_1 = M_2 = P\ell/4$

Region D: $M_1 = M_2 = Qh/4$

Region E: $M_1 = M_2 = (P\ell + Qh)/6$

Region F: $M_1 = (P\ell - Qh)/2$, $M_2 = Qh/2$

Region G: $M_1 = P\ell/2$, $M_2 = Qh/4$

$$W_A = c\left\{2\ell\left(\frac{P\ell}{4}\right)^{2/3} + 2h\left(\frac{Qh}{2}\right)^{2/3}\right\}, \qquad (12)$$

equal to the weight of the design (11),

$$W_B = c\left\{2\ell\left(\frac{P\ell}{2}\right)^{2/3} + 2h\left(\frac{Qh - P\ell}{2}\right)^{2/3}\right\}. \qquad (13)$$

The boundaries of all regions in Fig. 4 can be found in this manner. The regions E and G, for instance, correspond to the mechanism combinations e,f and d,f, respectively. From the condition (4) it is found that E cannot extend to the right of 11-12 and G cannot extend to the left of 11-13. By equating the structural weights of the designs E and G, the common boundary 11-14 of these regions is obtained.

Foulkes[8] has derived a similar design chart for $\alpha = 1$. This is shown in Fig. 5; the formulas specifying the minimum weight design for each region are the same as those given in the legend of Fig. 4. Comparison of Figs. 5 and 4 reveals that the change from $\alpha = 1$ to the more realistic value $\alpha = 2/3$

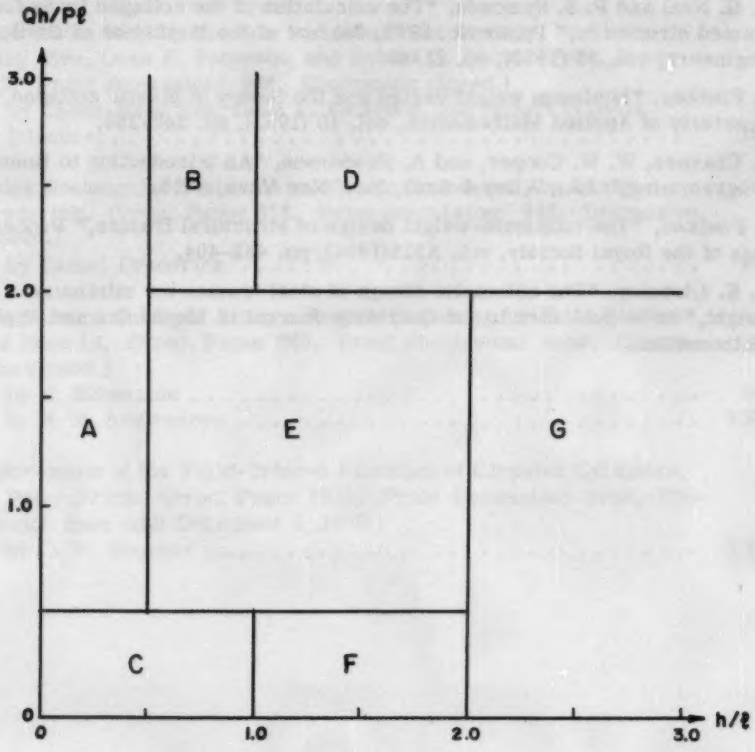

FIG. 5

does not change the general arrangement of the regions, though it introduces a considerable amount of distortion.

BIBLIOGRAPHY

1. J. F. Baker, "The design of steel frames," The Structural Engineer, vol. 27 (1949), pp. 397-431.

2. H. J. Greenberg and W. Prager, "Limit design of beams and frames," Proceedings, American Society of Civil Engineers, Sep. No. 59 (1951); Transactions, vol. 117 (1952), pp. 447-484 (with discussion).

3. D. C. Drucker, W. Prager, and H. J. Greenberg, "Extended limit design theorems for continuous media," Quarterly of Applied Mathematics, vol. 9 (1952), pp. 381-389.

4. J. Heyman, "Plastic design of beams and plane frames for minimum material consumption," Quarterly of Applied Mathematics, vol. 8 (1951), pp. 373-381; see also "Plastic design of plane frames for minimum weight," Structural Engineer, vol. 31 (1953), pp. 125-129.

5. B. G. Neal and P. S. Symonds, "The calculation of the collapse loads for framed structures," Paper No. 5772, Journal of the Institution of Civil Engineers, vol. 35 (1950), pp. 21-40.

6. J. Foulkes, "Minimum weight design and the theory of plastic collapse," Quarterly of Applied Mathematics, vol. 10 (1953), pp. 347-358.

7. A. Charnes, W. W. Cooper, and A. Henderson, "An introduction to linear programming," John Wiley & Sons, Inc. New York, 1953.

8. J. Foulkes, "The minimum-weight design of structural frames," Proceedings of the Royal Society, vol. A223(1954), pp. 482-494.

9. R. K. Livesley, "The automatic design of steel frames for minimum weight," to be published in the Quarterly Journal of Mechanics and Applied Mathematics.

Journal of the
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Proceedings of the American Society of Civil Engineers

CONTENTS

DISCUSSION
(Proc. Paper 1091)

Page

Stability of Beam Columns Above the Elastic Limit, by Robert L. Ketter. (Proc. Paper 692. Prior discussion: 819. Discussion closed. There will be no closure.)

Flow Into a Well by Electric and Membrane Analogy, by Chong-Hung Zee, Dean F. Peterson, and Robert O. Bock. (Proc. Paper 817. Prior discussion: 946. Discussion closed.)
by Chong-Hung Zee, Dean F. Peterson, and Robert O. Bock (closure) 1091-3

Thick Rectangular Plates on an Elastic Foundation, by Daniel Frederick. (Proc. Paper 818. Prior discussion: 946. Discussion closed.)
by Daniel Frederick 1091-7

The Viscous Sublayer Along a Smooth Boundary, by H. A. Einstein and Huon Li. (Proc. Paper 945. Prior discussion: none. Discussion closed.)
by E. Silberman 1091-9
by N. N. Ambraseys 1091-11

Suppression of the Fluid-Induced Vibration of Circular Cylinders, by Peter Price. (Proc. Paper 1030. Prior discussion: none. Discussion open until December 1, 1956.)
by G. W. Housner 1091-13

Note: Paper 1091 is part of the copyrighted Journal of the Engineering Mechanics Division of the American Society of Civil Engineers, Vol. 82, EM 4, October, 1956.

Discussion of "FLOW INTO A WELL BY ELECTRIC AND MEMBRANE ANALOGY"

by Chong-Hung Zee, Dean F. Peterson, and Robert O. Bock
(Proc. Paper 817)

CHONG-HUNG ZEE,[1] DEAN F. PETERSON, M. ASCE,[2] and ROBERT O. BOCK.[3]—The writers are gratified by the suggestions and comments of the discussers. Mr. Glover states that real wells are almost invariably in a transient state and rightly suggests that the transient state equations are presently incapable of handling the complications introduced in the mathematical treatment of unconfined aquifers by wells of relatively large drawdown. He outlines the conditions under which a steady state may be postulated without material error. Use of methods of analysis suggested by the writers and their associates(3) makes possible the use of a substitute steady state in order to overcome these difficulties.

The writers agree that curves in the form of Mr. Hansen's Eq. 3 would be better than those of Fig. 9. They do not think, however, that Mr. Hansen's derivation of this equation has complete theoretical validity. Mr. Hansen's Eq. 1 is believed to be valid. It contains five variables all having the dimension of length. Since only one fundamental dimension is involved, dimensional analysis can reduce these to no less than four, thus Hansen's Eq. 2 appears valid. Algebraic manipulation cannot reduce these variables to three as in Eq. 3. If Eq. 3 is valid, one of the variables of Hansen's Eq. 1 would have to be redundant or insignificant. There is some room for arbitrary choice of the fourth variable for Hansen's Eq. 3; however, h_w/r_w appears logical. This would lead to

$$\frac{h - h_w}{r_w} = f_3\left(\frac{r}{r_w}, \frac{h_s - h_w}{r_w}, \frac{h_w}{r_w}\right), \qquad \text{(A)}$$

instead of Hansen's Eq. 3.

As a practical matter, it is possible that h_w/r_w can be discarded. This is tantamount to discarding h_w in Hansen's Eq. 1. The test for validity must be an experimental one and an extension of the analysis begun by Mr. Hansen in his discussion is a logical approach. The writers regret that they have been unable to complete such an analysis; however, hydrodynamically, they believe all five of the variables of Hansen's Eq. 1 to be significant. The fact that the experimental data shown in Hansen's Fig. A correlate fairly well at $r/r_w = 115$ may indicate that one of the variables becomes relatively unimportant at this distance. The writers' Eq. 10 on which Fig. 9 is based follows from consideration of the membrane only and is independent of hydrodynamics as such. Its only relationship to the hydrodynamic case is through the validity of the membrane as an analogue.

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The introduction of dimensional analysis methods to well problems by Mr. Hansen and his use of the membrane to model the free surface(3) led to the combination of electrical and membrane analogies used by the writers for obtaining information regarding wells of large drawdown in unconfined aquifers.

Mr. Luthin raises the question of capillary effects and describes the complicated phenomena involved when such effects are considered. His statement regarding the relative importance of such effects in terms of specific field conditions will be useful in the application of well hydraulics to practical problems. One of the great obstacles to studying well hydraulics by utilizing simple scale models is the very presence of capillary effects which cannot be scaled. The membrane analogy eliminates these effects and is valid if capillary effects in the field are negligible. This is one of the principal advantages of the method as compared to the sand model.

Mr. Yih states that the membrane analogy is not a rigorously correct one because the second derivative of head with respect to z is not zero. This is true. On the other hand, the differential equation for the membrane is not rigorously correct if the drawdown is appreciable, because in deriving Eq. 5 one assumes that the slope length component ds is equal to its horizontal projection. A solution of the complete differential equations of elasticity for a circular membrane of finite unit weight and large central deflection is unknown to the writers. The similarity of Eq. 5 to Eq. 6 implies an approximate analogy which is borne out by the consistency of data obtained from its use with those from other sources, particularly from solutions using relaxation methods. Additional information by Boulton[1] closely agrees with the writer's results. Data for the four cases solved yield the following parameters:

Case No.	Q/Kr_w^2	h_s/r_w	$\frac{h_w}{r_w}$	h_{115}/r_w	$\frac{h_{115}-h_s}{r_w}$
1	9240	114.3	64	113.2	18.9
2	819	27.4	16	38.6	11.2
3	1010	25.8	0	39.0	13.2
4	18.2	1.5	0	5.2	3.7

These points have been plotted on Fig. 12. The drawdown curves associated with examples 1, 3 and 4 have been plotted in Fig. A.

If $\partial^2 h/\partial z^2$ in Eq. 6 be zero and if the weight of the membrane be ignored, for very small deflections the solution to both Eq. 5 and Eq. 6 is of the form of Glover's Eq. b, —linearly logarithmic in the radial direction. In this event, both problems may be quite easily handled by simple potential theory. The experimental data do not seem to substantiate a linear logarithmic distribution for either the well of large drawdown or the membrane of large deflection. One would not be surprised, if for correct boundary conditions, small deviations of the membrane from the true position of the free water surface made little difference in the discharge and seepage face; however, the

1. Boulton, Norman Savage. The flow pattern near a gravity well in a uniform water bearing medium. Jour. Inst. of Civ. Engr., London, Dec. 1951.

evidence substantiates that the membrane closely describes the true location of the free water surface also. The membrane analogy, in combination with the electric analogy, accordingly constitutes a new and useful tool for the solution of problems of flow to a well.

Corrections.—Eq. 3 should read

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2} + \frac{\partial^2 h}{\partial \theta^2} + \frac{\partial^2 h}{\partial z^2} = 0.$$

P. 10 - h_w in 4th line of last paragraph should read h_s.
P. 14 - Add the following sentence to the 4th paragraph:

At this time $Q/Kr_w^2 = 314$, $h_w/r_w = 13.9$,

$h_{115}/r_w = 26.6$.

Discussion of
"THICK RECTANGULAR PLATES ON AN ELASTIC FOUNDATION"

by Daniel Frederick
(Proc. Paper 818)

DANIEL FREDERICK,[1] J.M. ASCE.—The author wishes to express appreciation to Prof. Lorsch for his interesting and informative comments on the paper. For the example under discussion, it is of value to point out to the profession that for concrete or steel plates on soils where the usual values of kh^4/D fall in the range of 1/20 to 1/1000 that bending stresses and other results by the Reissner theory differ from those by the classical theory by a small amount. The value of $kh^4/D = 1$ will be significant for a concrete slab ($E = 2000$ ksi $\nu = 0.2$) on a rock foundation ($E = 3000$ ksi, $t = 10''$, then $k = 300$ kips per cubic inch), which gives a plate thickness of 0.581 inches.

In writing the paper, it was the writer's intent to show how the Reissner theory can be used in connection with plates on an elastic foundation and to present some solutions and methods of solving the governing equations. For this reason, no mention was made of any applications and it was felt that the readers would apply the corrections to their own particular problems where necessary. Values of kh^4/D in the range 1/10 to 1/1000 were used by the author in preparing calculations but were not presented in the final paper due to space limitations and the desire to have the corrections prominent in magnitude and direction.

Since many believe that the Winkler-Zimmerman assumption does not give a good approximation to the behavior of many soils, it is left to the reader's judgment to decide whether to use results based upon the classical elastic foundation. On the other hand there are many other practical problems for which kh^4/D is in the neighborhood of one and for which the secondary effects are important. Some of these are (a) thick blocks of ice floating on large bodies of water (b) a piston compressing a liquid in a cylinder (c) buckling and bending of thick layers of rock to explain certain phenomena in geophysics (d) plates with a low E resting on a relatively stiff elastic foundation. These might be made of plastics, resins, rubber, wood, etc. or metals at an elevated temperature where there has been a considerable decrease in Young's Modulus, and (e) bending of finite thick plates. It is important to point out that the infinite plate example of the paper does not bring in the ratio of thickness of the plate to the length. As in the finite plate with no foundation where the corrections to the classical theory are important for certain thickness-length ratios, it is expected the corrections would be considerable for finite plates of certain configurations and loadings on a foundation even for the range for kh^4/D of 1/1000 to 1/20.

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"THICK RECTANGULAR PLATES ON AN ELASTIC FOUNDATION"

by Dennis Fredericks
(Proc. Paper 2181)

DISCUSSION PERIOD.[1] 1.34 ASCE.—The author wishes to express appreciation to Prof. Lorenz for his interesting and informative comments on the paper. In the example under discussion, it is of value to point out to the reader that the exercise of trial plates to study where the point values of D/D_0 fall in the range of 1:20 to 1:1000 that Hetényi stresses and other results by the Timoshenko theory differ from those by the classical theory by a small amount. The value of D/D_0 = 1 will be significant for a concrete slab (E = 2000 ksi = 14,000,000 psf, μ = 0.15) on a rock foundation (k = 3000 psi, t = 10', then k = 300 kips per cubic inch), which gives a plate thickness of 0.861 inches.

In writing the paper, I was the writer's intent to show how the theory can be used to determine trial plates on an elastic foundation and to present some simple solutions and methods of solving the governing equations. For this reason, no limiting size scale of any application was given that the reader would apply the conclusions to their own particular problems where necessary. Values of D/D_0 in the range 1:10 to 1:1000 were used by the author to prepare the curves but were not presented in the final paper due to space limitations and the desire to have the corrections provided in engineering theory and discussion.

Prof. Lorenz believes that the Winkler-Zimmermann assumption does not give a good approximation to the behavior of many soils. It is left to the reader's judgment in every instance to use results based upon the classical elastic theorems. On the other hand there are many other practical problems for which D/D_0 is in the neighborhood of unity and for which the assumptions are important. Some of these are (a) thick floats or log floating on large bodies of water (b) a plate compressing a liquid in a cylinder (c) footing and loading of thick layers of rock to explain certain phenomena in geophysics (d) plates with a low E reading on a relatively stiff elastic foundation. These might be cases of planting, riveting, rubber, wood, or material at an elevated temperature where there has been a considerable increase in Young's Modulus and (e) floating of flexible thick plates. It is important to point out that the interior edge example of the paper does not imply in the ratio of thickness of the plate to the length, as in the finite plate with no foundation where the correction to the classical theory are important for certain thickness-length ratios, it is expected the corrections would be considerable for thick plates of certain configurations and loadings on a foundation even for the range for D/D_0 of 1:1000 to 1:20.

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Discussion of "THE VISCOUS SUBLAYER ALONG A SMOOTH BOUNDARY"

by H. A. Einstein and Huon Li
(Proc. Paper 945)

E. SILBERMAN,* A.M. ASCE.—The hypothesis of a periodic viscous sublayer in turbulent boundary layer flow is intriguing. Better experimental evidence of its existence than that offered in the paper is desirable. A possible experiment for direct visual observation of periodicity will be described subsequently.

First, the writer would like to suggest a modification in the theoretical development beginning with Section B of the paper, dealing with the definition of sublayer thickness. Instead of using displacement thickness, let the edge of the sublayer be given by

$$z = \delta \qquad \text{when } u/U_o = 0.99 \tag{6A}$$

Then, using Equations (3) and (4),

$$\delta = 3.64\sqrt{\nu t} \tag{8A}$$

(The choice of $u/U_o = 0.99$ is somewhat arbitrary; had $u/U_o = 0.995$ been used, the coefficient in the last equation would be 3.98 instead of 3.64.) Let the critical Reynolds number for sublayer instability be

$$\delta u_*/\nu\big|_{critical} = C \tag{15A}$$

where C is, presumably, a universal constant for boundary layer flow. Letting $\delta = \delta_T$ when $t = T$ and using Equation (14),

$$U_o/u_* = C\sqrt{\pi}/7.28 \tag{14A}$$

Equation (16) in the paper is taken from Nikuradse's equation

$$u/u_* = 5.75\ \mathrm{Log}_{10}(z u_*/\nu) + 5.5 \tag{16A}$$

This equation is determined empirically from the completely turbulent part of the flow near a smooth wall and is apparently universally applicable to all smooth pipes, channels and flat plates (with some room for argument about the exact values of the numerical constants). When $z = \delta_T$, $zu_*/\nu = C$ by the hypothesis of the paper and $u = 0.99\ U_o$. Inserting these values in Equation

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(16A) and combining with Equation (14A), C = 66 and $U_o/u_* = 16.1$. (Schlichting[9] gives C = 70 from inspection of Nikuradse's pipe data.)

The preceding argument arrives at essentially the same value of U_o/u_* as that found by the authors on the assumption that $\delta = 3\delta^*_T$, but with little question as to what thickness definition to use for the sublayer. (Had $u/U_o = 0.995$ been used, C = 73 and $U_o/u_* = 16.25$.)

There is another and remarkable conclusion from the above development; C = 66 — no boundary layer flow in which $\delta u_*/\nu < 66$ can be turbulent! This has far-reaching implications in determining the minimum Reynolds number for transition in flow in ducts or channels with smooth boundaries, but will not be developed further here. Using this criterion, it is possible to describe a flow in which the periodic viscous sublayer, if it exists, should be clearly visible.

Such an experiment would be performed on a fully developed turbulent flow in the vertex of a V-shaped channel or pipe with very small included angle. Here, measuring δ normally to the walls, $\delta u_*/\nu < 66$ until a considerable depth from the vertex has been reached, and the viscous sublayer should be stretched to finite dimensions. Some measurements with air flowing in a pipe of this shape with included angle of 12 degrees were described by Eckert and Irvine.[10] They did, in fact, find what they described as a very deep laminar flow region (see authors' statement in par. L2) and beyond that another relatively deep region with some indication of intermittent turbulence; but they were not seeking and did not observe periodicity in the flow. These, or similar experiments with dye streams in liquid, should be repeated with the objective of looking for periodicity. In such experiments, dye would have to be introduced through wall taps in order to avoid disturbing the stream. It is suggested that in a channel with small vertex angle at the bottom, dyes of variable density, slightly less than that of the flowing fluid, could be introduced at the vertex so that the dye streams would rise by buoyancy into the supposedly periodic part of the boundary layer for observation.

The writer believes that the most objectionable assumption in the development of the theory is the third assumption (p. 4): that of simultaneous growth of the sublayer over a large area of the boundary. It is this assumption which leads the authors to expect the period in the pressure auto-correlation at a point to be the same as the calculated period of the sublayer. It is more reasonable to expect the instability of the sublayer to be a function of both time and space as was, indeed, found by Schubauer and Klebanoff[11] for the laminar boundary layer in their investigation of transition. Under these circumstances, the authors, in their experiments, should have obtained pressure correlations from two points of variable separation along the stream direction, and looked for peaks in the correlation curve at intervals of KU_oT where $KU_o < U_o$ and represents the average rate of motion of the periodic part of the sublayer.

The writer's objection to the third assumption does not invalidate the

9. Schlichting, H. Boundary Layer Theory, McGraw Hill, New York, 1955, p. 407.
10. Eckert, E.R.G., and Irvine, T.F. "Simultaneous Turbulent and Laminar Flow in Ducts with Non-circular Cross Sections," Jour. Aero. Sci., Readers' Forum, January 1955, pp. 65-66.
11. Schubauer, G. B. and Klebanoff, P. S. "Contributions on the Mechanics of Boundary-Layer and Transition," NACA TN 3489, September 1955, 31 pp.

theory, however. For, using reasoning similar to that used by Schlichting in his discussion of periodic boundary layers,[12] Equations (1) and (2) may still be considered as a first approximation to the more complete equations of motion and boundary conditions.

N. N. AMBRASEYS,* J.M. ASCE.—The authors are to be congratulated upon their courage in venturing upon a subject that might well be described as intangible and elusive. The writer does not find anything in the paper to disagree with, and he only wished that experiments had allowed of treatment of some additional characteristics of the turbulent flow within the sub-layer, which should allow the writer to compare the hypothesis presented by the authors with results attained from similar experimental studies on the non-isotropic flow near the wall.

The writer would not have joined in the discussion but for the fact that for some time he had been engaged with this very subject.** His study is temporarily interrupted, so that the writer prefers not to present any of the results attained until now, but to wait till the study of the problem is completely finished.

However, briefly one or two of the consequences of the results attained so are, will be mentioned.

i) The thickness of the sub-layer, defined as the thickness of the zone within which the velocity distribution does not obey von Kàrmàn's law, was found to extend to a dimensionless distance from the wall, equal to 26.6 to 30.1, according to the assigned values for the universal coefficients κ and β (κ found to vary from 0.33 to 0.4, and β from 0.111 to 0.113).

ii) Scales and microscales of turbulence were found to exhibit a strong dependance on the amount of the local mean velocity within the sub-layer.

iii) It was found that the part of energy produced within the sub-layer by the pressure drop, and converted into turbulent energy, was influenced by basic flow characteristics, but mainly by the amount of mean velocity available in these parts. So, length scales in the sub-layer are not position functions, but depend greatly on the local mean values of the velocity. This proves that, actually the "mixing length" hypothesis does not apply for the quasiturbulent region near the wall.

iv) A very simple expression was found to connect turbulent shear and local mean velocity, for values of the dimensionless distance less than 27, and a new universal constant was defined.

The writer would like to learn from the authors whether they have observed any dependance of the turbulent shear, on the mean velocity. The point is of vital interest in its possible bearing on the time average values of mean velocities and the period T.

12. Loc. cit., pp. 180-181.

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** Hydr. Research 1954. International Assn. for Hydr. Research, p. 132.

Discussion of "SUPPRESSION OF THE FLUID-INDUCED VIBRATION OF CIRCULAR CYLINDERS"

by Peter Price
(Proc. Paper 1030)

G. W. HOUSNER,* A.M. ASCE.—The data presented in this paper are very informative, and it is hoped that such additional experiments, aimed at the civil engineering aspects of this aerodynamic problem, will be reported in the literature. More research and study with wind tunnel models and with prototype stacks is needed to clarify the aerodynamic features of the problem.

The data presented by the author for supercritical flow, in his Figure 7, are particularly interesting because they show behavior quite different from that obtained if periodic vortex shedding occurs at a Strouhal number S_n = 0.2. The air velocities in the experiment were such that vortex shedding at S_n = 0.2 would give frequencies greater than the natural frequency of the model whereas the vibrations observed, although erratic, always occurred with the natural frequency of the model. Since wind tunnel experiments show that above the critical velocity the aerodynamic forces become aperiodic it seems likely that the observed vibrations were produced by aperiodic forces. A simple approximate analysis can be made which shows that the data are consistent with vibrations produced by aperiodic forces.

If the simple oscillator shown in the writer's Figure 1 is acted upon by a sinusoidal force F sin 2π ft, the maximum displacement at resonance is

$$y_{max} = \frac{F}{k} \quad \frac{1}{2c} \quad = y_{st} \quad \frac{1}{2c}$$

where k is the spring constant (F/k = static deflection) and 1/2c is the magnification factor, c being the fraction of critical damping ($c = \delta/2\pi$, for small damping, δ = logarithmic decrement). On the other hand, if F is an intermittent force that produces impulses $I = F\Delta t$ that are aperiodic, or random in time, the behavior of the oscillator is quite different. Suppose that at the instant the oscillator has velocity v it receives an impulse I; from impulse-momentum relations this produces a change in velocity $\Delta v = \frac{I}{m}$. The change in kinetic energy is

$$\Delta E = \frac{1}{2} m(v + \Delta v)^2 - \frac{1}{2} m v^2$$

$$= mv\,\Delta v + \frac{1}{2} m(\overline{\Delta v})^2$$

If the oscillator is subjected to N impulses the total energy input will be

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$$E_i = \sum_N mv\,\Delta v + \sum_N \frac{1}{2} m(\Delta v)^2$$

Since the direction and timing of the impulses are random, on the average the first summation will cancel out and E_i is given by the second term, that is

$$E_i = N \frac{1}{2} \frac{I}{m}^2$$

The energy dissipated per cycle is approximately $4\pi cE$, where E is the total energy of the oscillator. Thus if n is the average number of impulses per second and f_n is the natural frequency of the oscillator, the average rate of change of energy is given by

$$\frac{\partial E}{\partial t} = n \frac{1}{2} \frac{I}{m}^2 - f_n 4\pi cE$$

When the oscillator starts from rest the solution of this equation is

$$E = \frac{nI^2}{8\pi c f_n m} \left(1 - e^{-f_n 4\pi ct}\right)$$

and so E increases to

$$E = \frac{nI^2}{8\pi c f_n m}$$

which corresponds to an amplitude

$$y_{max} = \sqrt{\frac{2E}{k}} = \frac{I}{2}\left(\frac{n}{k\pi c f_n m}\right)^{1/2} = \frac{I}{k} f_n \left(\frac{n\pi}{f_n c}\right)^{1/2}$$

and the vibrations will have essentially the natural frequency of the model.

To adapt this to the present case, consider the wind tunnel model to be an oscillator as shown in the writer's Figure 1 with the following properties.

$k = 18{,}000$ lbs per ft.
$c = 0.005$
$\ell = 45''$
$f_n = 37.5$ cps

In the subcritical range with vortices shed at a frequency

$$f_v = 0.2 \frac{v}{D}$$

let the effective force on the oscillator be

$$F = c_L q D \frac{\ell}{4} \sin 2\pi f_n t$$

$$q = \frac{1}{2}\rho v^2$$

During one-half cycle at resonance F exerts an impulse $I = F/\pi f_n$ so the maximum displacement is

$$y_r = \frac{F}{k\,2c} = \frac{\pi f_n I}{k\,2c}$$

In the supercritical region, let it be assumed that the aperiodic impulses occur with

$$n = 2 f_v$$

$$I = F/\pi f_n$$

$$F = \alpha c_L q \frac{\ell}{4}$$

From the ratio of y_{max} to y_r it is seen that

$$y_{max} = y_r \left(\frac{8c}{\pi}\frac{f_v}{f_n}\right)^{1/2} = \frac{F}{2ck}\left(\frac{8c}{\pi}\frac{f_v}{f_n}\right)^{1/2} = \frac{\pi f_n I}{2c\,k}\left(\frac{8c}{\pi}\frac{f_v}{f_n}\right)^{1/2}$$

When y_{max} is evaluated for the following values of parameters

$c = 0.005$
$k = 18{,}000$ lb per ft
$\alpha = 0.75$
$c_L = 1/2$
$\ell = 45''$
$D = 6''$

and the curve is plotted in the same form as the author's Figure 7, the result is as shown in the writer's Figure 2 where it is seen that there is reasonable agreement with the experimental data. It should be noted that the foregoing analysis is only approximate, for the details of the aerodynamics are not known, nor are the precise values of the model parameters known. The analysis only shows that the experimental data are not inconsistent with the hypothesis of vibrations produced by aperiodic forces.

It is of interest to note that y_{max} is smaller than y_r by the factor

$$\alpha \left(\frac{8c}{\pi}\frac{f_v}{f_n}\right)^{1/2}$$

For $c = 0.005$ and $f_v/f_n = 1$ this is equal to 0.11α so that the vibrations induced by the aperiodic forces have an amplitude approximately 1/10 as large as would be induced by periodically shed vortices at resonance. This may be compared with some field measurements made at the writer's request on a

steel stack of 85 ft cantilever, 4'-0" diameter, 1/4" wall thickness, and 1.3 cps natural frequency. The maximum oscillations of the stack were observed at a wind velocity of 26 fps (Reynolds number = 700,000) which corresponds to resonance at a Strouhal number of 0.22. The maximum tip displacement was 3" which, assuming periodic vortex shedding, corresponds to a magnification factor of approximately 50 and a fraction of critical damping c = 0.01 which is reasonable for an unlined stack. At 40 fps wind velocity the maximum observed tip amplitude was 1.5". Since the author's experimental data would indicate a magnification factor of approximately 5 instead of 50 and would also indicate a larger amplitude at 40 fps than at 26 fps it would seem that there were different types of aerodynamic flow patterns around the stack and in the wind tunnel. It might also be pointed out that at a wind velocity of 18 fps the stack was observed to vibrate at a frequency of 1.1 cps which lies between the 0.9 cps predicted by $S_n = 0.2$ and the natural frequency, 1.3 cps, of the stack. The same frequency modification was noted by the author in his subcritical velocity experiments.

If the author's data and the field observations are accepted, the conclusion would seem to be that the designer must consider two possibilities. First, occasional resonant vibrations may occur at a Strouhal number of 0.2, which would give a maximum design condition. Second, when the wind velocity reaches a value approximately four times the resonant velocity (depending on damping) the amplitudes of vibration produced occasionally by aperiodic forces may be of the order of magnitude of the resonant amplitudes. However, additional experimental data are required before these conclusions could be accepted. It is hoped that in his closure the author will explain the influence of damping on the data presented in his Figures 3 and 5. If the frequency of vibration is the same at resonance for each of the test runs and the hydrodynamic forces and the fraction of critical damping are also the same the maximum amplitude must vary inversely proportional to k, which means that the curve in Figure 3 should be a straight line and the product of the double amplitude times k should be a constant. This product is approximately 400 for runs 8 through 11 but is only 50 for run 7. It may be that run 7 was governed chiefly by the flow pattern whereas runs 8-11 were governed largely by damping. From the author's description it appears that the damping in the model was very small (order of c = 0.002) with consequent large magnification factors. The amplitude would then be very sensitive to small changes in fraction of critical damping. The experimental procedure of changing I while keeping I/k constant could make a large difference in the fraction of critical damping and hence a large difference in magnification factor. It is hoped that the author will discuss this point and will also explain how much of the beneficial effect of the shroud shown in Figure 5 was produced by increase in damping. Until the matter of the damping is settled, the data shown in Figures 3 and 5 cannot be interpreted.

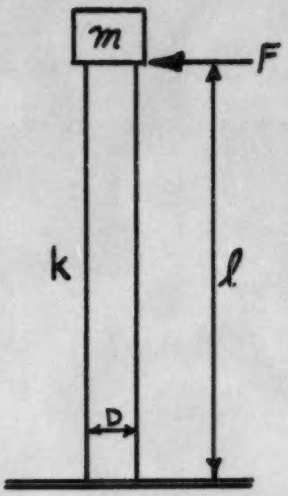

FIGURE 1

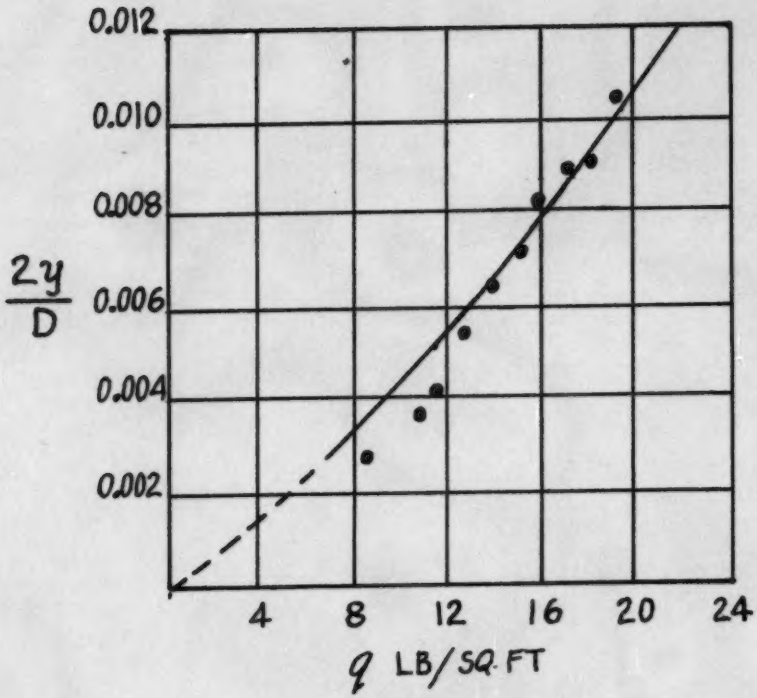

FIGURE 2